A logarithmic generalization of tensor product theory for modules for a vertex operator algebra

Yi-Zhi Huang, James Lepowsky and Lin Zhang

Abstract

We describe a logarithmic tensor product theory for certain module categories for a "conformal vertex algebra." In this theory, which is a natural, although intricate, generalization of earlier work of Huang and Lepowsky, we do not require the module categories to be semisimple, and we accommodate modules with generalized weight spaces. The corresponding intertwining operators contain logarithms of the variables.

1 Introduction

In this paper, we present a generalization of the tensor product theory developed by the first two authors in [HL1]–[HL5], [H1] and [H2] to a logarithmic tensor product theory for certain module categories for a "conformal vertex algebra." We focus on explaining new results and sophisticated techniques in this logarithmic theory, rather than on the full details of the proofs, which are being presented in a longer paper [HLZ].

Logarithmic structure in conformal field theory was first introduced by physicists to describe disorder phenomena [Gu]. A lot of progress has been made by physicists on this subject. We refer the interested reader to review articles [Ga], [F] and [RT] (and references therein). Such structures also arise naturally in the representation theory of vertex operator algebras. In fact, in the construction of intertwining operator algebras, the first author proved (see [H2]) that if modules for the vertex operator algebra satisfy a certain cofiniteness condition, then products of the usual intertwining operators satisfy certain systems of differential equations with regular singular points. In addition, it was proved in [H2] that if the vertex operator algebra satisfies certain finite reductivity conditions, then the analytic extensions of products of the usual intertwining operators have no logarithmic terms.

In the case when the vertex operator algebra satisfies only the cofiniteness condition but not the finite reductivity conditions, the products of intertwining operators still satisfy systems of differential equations with regular singular points. But in this case, the analytic extensions of such products of intertwining

operators might have logarithmic terms. This means that if we want to generalize the results in [HL1]–[HL5], [H1] and [H2] to the case in which the finite reductivity properties are not always satisfied, we have to consider intertwining operators involving logarithmic terms.

In [M], Milas introduced and studied what he called "logarithmic modules" and "logarithmic intertwining operators." Roughly speaking, logarithmic modules are weak modules for vertex operator algebras that are direct sums of generalized eigenspaces for the operator $L(0)$ (we will start to call such weak modules "generalized modules" in this paper), and logarithmic intertwining operators are operators which depend not only on powers of a (formal or complex) variable x, but also on its logarithm $\log x$.

In [HLZ], we have generalized the tensor product theory developed by the first two authors to the category of generalized modules for a "conformal vertex algebra", or even more generally, for a "Möbius vertex algebra", satisfying suitable conditions. The special features of the logarithm function make the logarithmic theory very subtle and interesting. Although we have shown that all the main theorems in the original tensor product theory developed by the first two authors still hold in the logarithmic theory, many of the proofs and techniques being used involve certain new and sophisticated techniques and have surprising connections with certain combinatorial identities.

In fact, one of the main technical tools that we use in this paper (and indeed, must use) is the same as a certain critical and subtle ingredient of the original tensor product theory of Huang and Lepowsky: a condition called the "compatibility condition" in [HL1]–[HL5] and [H1]. In the construction of the associativity isomorphism in [H1], one of the important steps is to identify the "intermediate module," which is the tensor product module of two modules. The compatibility condition is the most important condition among those conditions characterizing the tensor product module and thus it is natural to use it to identify the intermediate module. (In fact, in a recent paper [HLLZ] we have proved by means of suitable counterexamples that the compatibility condition is required in the theory.) The method based on the compatibility condition depends on the simple but powerful result that any "intertwining map" determines an intertwining operator uniquely. To generalize the original theory to the logarithmic theory, one first has to generalize this result, and the generalization of this basic result involves some subtle properties of the logarithmic function. In particular, we see that the method based on the compatibility condition, and therefore all the proofs in the logarithmic theory, are more delicate than in the original theory. In the present paper, we shall explain the results obtained in this new logarithmic theory. Further details and proofs are contained in [HLZ].

One important application of this generalization is to the category $\mathcal{O}_\kappa$ of certain modules for an affine Lie algebra studied by Kazhdan and Lusztig in their series of papers [KL1]–[KL5]. It has been shown in [Z] by the third author that, among other things, all the conditions needed to apply our theory to this module category are satisfied. As a result, it is proved in [Z] that there is a natural vertex tensor category structure on this module category, giving a new construction, in the context of general vertex operator algebra theory, of the

braided tensor category structure on $\mathcal{O}_\kappa$. The methods used in [KL1]–[KL5] were very different.

The present work is a natural (although subtle) extension of [HL1]–[HL5], [H1] and [H2]. This work was presented by one of us (L.Z.) in seminars at Rutgers and at Stony Brook in April, 2003.

Acknowledgment The authors would like to thank Sasha Kirillov Jr., Masahiko Miyamoto, Kiyokazu Nagatomo and Bin Zhang for their early interest in our work on logarithmic tensor product theory, and Bin Zhang in particular for inviting L.Z. to lecture on this ongoing work at Stony Brook. Y.-Z.H. would also like to thank Paul Fendley for raising his interest in logarithmic conformal field theory. The authors gratefully acknowledge partial support from NSF grants DMS-0070800 and DMS-0401302.

2 The setting

In this section we define some notions that will be used in this paper. We shall assume the reader has some familiarity with the material in [B], [FLM] and [FHL], such as the formal delta function, the binomial expansion convention, the notion of vertex (operator) algebra, and the notion of module for a vertex (operator) algebra. We also refer the reader to [LL] for such material.

Throughout this paper we use the following usual conventional notation: the symbols x, x_0, x_1, x_2, $\dots$, y, y_0, y_1, y_2, $\dots$ will denote commuting independent formal variables, and by contrast, the symbols z, z_0, z_1, z_2, $\dots$ will denote complex numbers in specified domains. We use the notation $\mathbb{N}$ for the nonnegative integers and $\mathbb{Z}_+$ for the positive integers.

We will use the following version of the notion of "conformal vertex algebra": A *conformal vertex algebra* is a vertex algebra equipped with a $\mathbb{Z}$-grading and with a conformal vector ω satisfying the usual conditions. Hence the only difference between a conformal vertex algebra and a vertex operator algebra is that a vertex operator algebra further satisfies two "grading restriction conditions" (see [FLM] and [FHL]).

Remark 2.1 Analogous to the notion of "quasi-vertex operator algebra" in [FHL], a slightly more general notion of "Möbius vertex algebra" can be defined: A *Möbius vertex algebra* is a vertex algebra equipped with a representation of $\mathfrak{sl}(2)$ on V satisfying the same conditions as those for the operators $L(-1)$, $L(0)$ and $L(1)$ in the definition of vertex operator algebra. Essentially all the results we shall describe in the present paper for conformal vertex algebras also hold more generally for Möbius vertex algebras. However, for brevity, in the present paper we shall state our results only for conformal vertex algebras.

As expected, a *module* for a conformal vertex algebra V is defined to be a module for V viewed as a vertex algebra such that the conformal element acts in the same way as in the definition of vertex operator algebra.

In this paper we will study a larger family of "modules" which satisfy all conditions for being a module except that they are only direct sums of *generalized* eigenspaces, rather than eigenspaces, for the operator $L(0)$. We call these *generalized modules*.

We also have the obvious notions of *module homomorphism*, *submodule*, *quotient module*, and so on.

For a generalized module W, we will denote by $W_{[n]}$ the generalized eigenspace for $L(0)$ with respect to eigenvalue n; we call n the corresponding *(generalized) weight*. Thus

$$W = \coprod_{n\in\mathbb{C}} W_{[n]};$$

and for $n \in \mathbb{C}$,

$$W_{[n]} = \{w \in W \mid (L(0) - n)^m w = 0 \text{ for } m \in \mathbb{N} \text{ sufficiently large}\}.$$

As in [FHL], from the $L(-1)$-derivative property and the Jacobi identity for generalized modules for a conformal vertex algebra V, we have

$$[L(0), v_n] = (L(0)v)_n + (-n - 1)v_n,$$

as operators acting on a generalized V-module, for any $v \in V$ and $n \in \mathbb{C}$. From this we obtain:

Proposition 2.2 *Let W be a generalized module for a conformal vertex algebra V. Let $L(0)_s \in End\ W$ be the "semisimple part" of the operator $L(0)$:*

$$L(0)_s w = nw \quad \textit{for}\ \ w \in W_{[n]},\ n \in \mathbb{C}.$$

Then the "locally nilpotent part" $L(0) - L(0)_s$ of $L(0)$ commutes with the action of V on W. In other words, $L(0) - L(0)_s$ is a V-homomorphism from W to itself.

Let W be a generalized module for a conformal vertex algebra. We will use the notation $\overline{W}$ for the completion of W with respect to the $\mathbb{C}$-grading, that is,

$$\overline{W} = \prod_{n\in\mathbb{C}} W_{[n]}.$$

We write

$$\pi_n : \overline{W} \to W_{[n]} \tag{2.1}$$

for the natural projection map from $\overline{W}$ to $W_{[n]}$, for any $n \in \mathbb{C}$. We will also use the same notation for its restriction to W, and for its natural extensions to spaces of formal series with coefficients in W.

Clearly there is a canonical pairing between $\overline{W}$ and the subspace $\coprod_{n\in\mathbb{C}}(W_{[n]})^*$ of W^* (viewing $(W_{[n]})^*$ as embedded in W^* in the natural way). Here and throughout this paper we use the notation M^* for the dual of a vector space M. We will also use $\langle\cdot,\cdot\rangle$ for the natural pairings between various vector spaces.

The following notion of "strong gradedness" for a conformal vertex algebra will be needed:

Definition 2.3 Let A be an abelian group. A conformal vertex algebra

$$V = \coprod_{n\in\mathbb{Z}} V_{(n)}$$

is said to be *strongly A-graded*, or just *strongly graded* if A is implicitly understood, if there is a second gradation on V, by A,

$$V = \coprod_{\alpha\in A} V^{(\alpha)},$$

such that the following conditions are satisfied: the two gradations are compatible, that is,

$$V^{(\alpha)} = \coprod_{n\in\mathbb{Z}} V^{(\alpha)}_{(n)} \quad (\text{where } V^{(\alpha)}_{(n)} = V_{(n)} \bigcap V^{(\alpha)}) \quad \text{for any} \quad \alpha \in A;$$

for any $\alpha, \beta \in A$ and $n \in \mathbb{Z}$,

$$V^{(\alpha)}_{(n)} = 0 \quad \text{for} \quad n \quad \text{sufficiently negative};$$
$$\dim V^{(\alpha)}_{(n)} < \infty;$$
$$\mathbf{1} \in V^{(0)}_{(0)};$$
$$v_l V^{(\beta)} \subset V^{(\alpha+\beta)} \quad \text{for any} \quad v \in V^{(\alpha)},\ l \in \mathbb{Z};$$

and

$$\omega \in V^{(0)}_{(2)}.$$

Given a strongly A-graded conformal vertex algebra V and an abelian group $\tilde{A}$ containing A as a subgroup, a (generalized) V-module W is said to be *strongly $\tilde{A}$-graded*, or just *strongly graded* if $\tilde{A}$ is implicitly understood, if the obvious analogues of the conditions for W in Definition 2.3 hold, except that the grading-truncation condition becomes: for any $\beta \in \tilde{A}$ and $n \in \mathbb{C}$,

$$W^{(\beta)}_{[n+k]} = 0 \quad \text{for} \quad k \in \mathbb{Z} \quad \text{sufficiently negative.} \tag{2.2}$$

Remark 2.4 Clearly, vertex operator algebras and their (ordinary) modules are exactly the conformal vertex algebras and their (ordinary) modules that are strongly graded with respect to the trivial group (with (2.2) used as the grading-truncation condition). Important examples of other strongly graded conformal vertex algebras and modules come from vertex algebras and modules associated with an even lattice that is not necessarily positive definite. We refer the reader to [Z] for the application of the results in the present work to these examples.

Suppose that $L(1)$ acts locally nilpotently on a conformal vertex algebra V (which occurs, in particular, if V is strongly graded). Then generalizing formula

(3.20) in [HL3] we define the *opposite vertex operator* on a generalized V-module (W, Y_W) associated to $v \in V$ by

$$Y_W^o(v,x) = \sum_{n\in\mathbb{Z}} v_n^o x^{-n-1} = Y_W(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}), \tag{2.3}$$

so that

$$v_n^o = (-1)^k \sum_{m\in\mathbb{N}} \frac{1}{m!}(L(1)^m v)_{-n-m-2+2k}$$

for $v \in V_{(k)}$ and $n, k \in \mathbb{Z}$. (Note that the $L(1)$-local nilpotence is needed for well-definedness here.) We have

$$v_n^o W_{[m]} \subset W_{[m+n+1-k]} \quad \text{for any} \quad m \in \mathbb{C},$$

and as in the case of [HL3], the opposite Jacobi identity for Y_W^o, as in formula (3.23) of [HL3], holds.

As in Section 5.2 of [FHL], we can define a V-action on W^* as follows:

$$\langle Y'(v,x)w', w\rangle = \langle w', Y_W^o(v,x)w\rangle \tag{2.4}$$

for $v \in V$, $w' \in W^*$ and $w \in W$; the correspondence $v \mapsto Y'(v,x)$ is a linear map from V to $(\operatorname{End} W^*)[[x, x^{-1}]]$. Define the operators $L'(n)$ $(n \in \mathbb{Z})$ by

$$Y'(\omega, x) = \sum_{n\in\mathbb{Z}} L'(n)x^{-n-2}.$$

Now we can introduce the notion of "contragredient module" of a strongly graded generalized module. First we give:

Definition 2.5 Let $W = \coprod_{\beta\in\tilde{A},\, n\in\mathbb{C}} W_{[n]}^{(\beta)}$ be a strongly $\tilde{A}$-graded generalized module for a strongly A-graded conformal vertex algebra. We define W' to be the $(\tilde{A}\times\mathbb{C})$-graded vector space

$$W' = \coprod_{\beta\in\tilde{A},\, n\in\mathbb{C}} (W')_{[n]}^{(\beta)}, \quad \text{where} \quad (W')_{[n]}^{(\beta)} = (W_{[n]}^{(-\beta)})^*; \tag{2.5}$$

also define

$$(W')^{(\beta)} = \coprod_{n\in\mathbb{C}} (W_{[n]}^{(-\beta)})^* \subset (W^{(-\beta)})^*$$

and

$$(W')_{[n]} = \coprod_{\beta\in\tilde{A}} (W_{[n]}^{(-\beta)})^* \subset (W_{[n]})^*,$$

the homogeneous subspaces of W' with respect to the $\tilde{A}$- and $\mathbb{C}$-grading, respectively. There is of course a canonical pairing between W' and $\overline{W} \subset \prod_{\beta\in\tilde{A},\, n\in\mathbb{C}} W_{[n]}^{(\beta)}$.

Let W and W' be as in this definition. It is straightforward to show that the lower truncation condition for Y' on W' holds; thus the Jacobi identity can be formulated on W'. Furthermore, using the same arguments as in the proofs of Theorems 5.2.1 and 5.3.1 in [FHL] we have:

Theorem 2.6 *Let $\tilde{A}$ be an abelian group containing A as a subgroup and V a strongly A-graded conformal vertex algebra. Let (W, Y) be a strongly $\tilde{A}$-graded V-module (respectively, generalized V-module). Then the pair (W', Y') carries a strongly $\tilde{A}$-graded V-module (respectively, generalized V-module) structure, and $(W'', Y'') = (W, Y)$.*

The pair (W', Y') in Theorem 2.6 will be called the *contragredient module* of (W, Y).

Throughout this paper, we will fix a conformal vertex algebra V. Given abelian groups A and $\tilde{A} \supset A$ as above, when V is strongly A-graded we will write $\mathcal{C}_1$ for the category of all strongly $\tilde{A}$-graded generalized V-modules. We have a contravariant functor $(\cdot)' : (W, Y) \mapsto (W', Y')$ from $\mathcal{C}_1$ to itself, which we call the *contragredient functor*. Our main objects of study will be certain full subcategories $\mathcal{C}$ of $\mathcal{C}_1$ that are closed under the contragredient functor.

3 Logarithmic intertwining operators

In this section we first introduce the "logarithm of a formal variable" and study some of its properties. We then study the notion of "logarithmic intertwining operator" introduced in [M] and give some results which are essential for our generalization of the tensor product theory.

We will use the following usual notation for formal series with arbitrary complex powers, as in [FLM]:

$$\mathcal{W}\{x\} = \left\{ \sum_{n\in\mathbb{C}} w(n)x^n \mid w(n) \in \mathcal{W},\ n \in \mathbb{C} \right\}$$

for any vector space $\mathcal{W}$ and formal variable x.

From now on we will sometimes need and use new independent (commuting) formal variables denoted by $\log x, \log y, \log x_1, \log x_2, \ldots$, etc. We will work with formal series in such formal variables together with the "usual" formal variables x, y, x_1, $x_2, \ldots$, etc., with coefficients in certain vector spaces, and the powers of the monomials in *all* the variables can be arbitrary complex numbers. (Later we will restrict our attention to only nonnegative integral powers of the "log" variables.)

Given a formal variable x, we write $\frac{d}{dx}$ for the linear map on $\mathcal{W}\{x, \log x\}$, for any vector space $\mathcal{W}$ not involving x, defined (and indeed well defined) by the (expected) formula

$$\begin{aligned}
&\frac{d}{dx}\Bigg(\sum_{m,n\in\mathbb{C}} w_{n,m}x^n(\log x)^m \Bigg) \\
&\quad = \sum_{m,n\in\mathbb{C}} ((n+1)w_{n+1,m} + (m+1)w_{n+1,m+1})x^n(\log x)^m \\
&\quad \Bigg(= \sum_{m,n\in\mathbb{C}} nw_{n,m}x^{n-1}(\log x)^m + \sum_{m,n\in\mathbb{C}} mw_{n,m}x^{n-1}(\log x)^{m-1} \Bigg)
\end{aligned}$$

where $w_{n,m} \in \mathcal{W}$ for all $m, n \in \mathbb{C}$. The same notation will also be used for the restriction of $\frac{d}{dx}$ to any subspace of $\mathcal{W}\{x, \log x\}$ that is closed under $\frac{d}{dx}$, e.g., $\mathcal{W}\{x\}[[\log x]]$ or $\mathbb{C}[x, x^{-1}, \log x]$. Clearly, $\frac{d}{dx}$ acting on $\mathcal{W}\{x\}$ coincides with the usual formal derivative.

Remark 3.1 Let f, g and f_i, i in some index set I, all be of the form

$$\sum_{m,n\in\mathbb{C}} w_{n,m} x^n (\log x)^m \in \mathcal{W}\{x, \log x\}, \quad w_{n,m} \in \mathcal{W}. \tag{3.1}$$

One checks the following straightforwardly: Suppose that the sum of f_i, $i \in I$, exists (in the obvious sense). Then the sum of $\frac{d}{dx} f_i$, $i \in I$, also exists and is equal to $\frac{d}{dx} \sum_{i\in I} f_i$. More generally, for any $T = p(x)\frac{d}{dx}$, $p(x) \in \mathbb{C}[x, x^{-1}]$, the sum of Tf_i, $i \in I$, exists and is equal to $T \sum_{i\in I} f_i$. Thus the sum of $e^{yT} f_i$, $i \in I$, exists and is equal to $e^{yT} \sum_{i\in I} f_i$ (e^{yT} being the formal exponential series, as usual). Suppose that $\mathcal{W}$ is an (associative) algebra or that the coefficients of either f or g are complex numbers. If the product of f and g exists, then the product of $\frac{d}{dx} f$ and g and the product of f and $\frac{d}{dx} g$ both exist, and $\frac{d}{dx}(fg) = (\frac{d}{dx} f)g + f(\frac{d}{dx} g)$. Furthermore, for any T as before, the product of Tf and g and the product of f and Tg both exist, and $T(fg) = (Tf)g + f(Tg)$. In addition, the product of $e^{yT} f$ and $e^{yT} g$ exists and is equal to $e^{yT}(fg)$, just as in formulas (8.2.6)–(8.2.10) of [FLM]. The point here, of course, is just the formal derivation property of $\frac{d}{dx}$, except that sums and products of expressions do not exist in general.

Remark 3.2 Note that the "equality" $x = e^{\log x}$ does not hold, since the left-hand side is a formal variable, while the right-hand side is a formal series in another formal variable. In fact, this formula should not be assumed, since, for example, the formal delta function $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$ would not exist in the sense of formal calculus, if x were allowed to be replaced by the formal series $e^{\log x}$. By contrast, note that the equality

$$\log e^x = x \tag{3.2}$$

does indeed hold. This is because the formal series e^x is of the form $1+X$ where X involves only positive integral powers of x and in (3.2), "log" refers to the usual formal logarithmic series

$$\log(1 + X) = \sum_{i\geq 1} \frac{(-1)^{i-1}}{i} X^i,$$

not to the "log" of a formal variable. We will use the symbol "log" in both ways, and the meaning will be clear in context.

We will typically use notations of the form $f(x)$, instead of $f(x, \log x)$, to denote elements of $\mathcal{W}\{x, \log x\}$ for some vector space $\mathcal{W}$ as above. For this reason, we need to interpret the meaning of symbols such as $f(x + y)$, or more generally, symbols obtained by replacing x in $f(x)$ by something other than

just a single formal variable (since $\log x$ is a formal variable and not the image of some operator acting on x). Specifically, we use the following notational conventions; the existence of the expressions will be justified in Remark 3.4:

Notation 3.3 For formal variables x, y, and $f(x)$ of the form (3.1), define

$$\begin{aligned} f(x+y) &= \sum_{m,n\in\mathbb{C}} w_{n,m}(x+y)^n\left(\log x+\log\left(1+\frac{y}{x}\right)\right)^m \\ &= \sum_{m,n\in\mathbb{C}} w_{n,m}(x+y)^n\left(\log x+\sum_{i\geq 1}\frac{(-1)^{i-1}}{i}\left(\frac{y}{x}\right)^i\right)^m; \end{aligned} \tag{3.3}$$

in the right-hand side, $(\log x+\sum_{i\geq 1}\frac{(-1)^{i-1}}{i}(\frac{y}{x})^i)^m$, according to the binomial expansion convention, is to be expanded in nonnegative integral powers of the second summand $\sum_{i\geq 1}\frac{(-1)^{i-1}}{i}(\frac{y}{x})^i$, so the right-hand side of (3.3) is equal to

$$\sum_{m,n\in\mathbb{C}} w_{n,m}(x+y)^n\sum_{j\in\mathbb{N}}\binom{m}{j}(\log x)^{m-j}\left(\sum_{i\geq 1}\frac{(-1)^{i-1}}{i}\left(\frac{y}{x}\right)^i\right)^j \tag{3.4}$$

when expanded one step further. Also define

$$f(xe^y)=\sum_{m,n\in\mathbb{C}} w_{n,m}x^n e^{ny}(\log x+y)^m, \tag{3.5}$$

$$f(xy)=\sum_{m,n\in\mathbb{C}} w_{n,m}x^n y^n(\log x+\log y)^m. \tag{3.6}$$

Remark 3.4 The existence of the right-hand side of (3.3), or (3.4), can be seen by writing $(x+y)^n$ as $x^n(1+\frac{y}{x})^n$ and observing that

$$\left(\sum_{i\geq 1}\frac{(-1)^{i-1}}{i}\left(\frac{y}{x}\right)^i\right)^j\in\left(\frac{y}{x}\right)^j\mathbb{C}\left[\left[\frac{y}{x}\right]\right].$$

The existence of the right-hand sides of (3.5) and of (3.6) is clear. Furthermore, both $f(x+y)$ and $f(xe^y)$ lie in $\mathcal{W}\{x,\log x\}[[y]]$, while $f(xy)$ lies in $\mathcal{W}\{xy,\log x\}[[\log y]]$.

We still have the following identities (cf. Section 8.3 of [FLM]) in the logarithmic settings:

Proposition 3.5 *For $f(x)$ as in (3.1), we have*

$$e^{y\frac{d}{dx}}f(x)=f(x+y) \tag{3.7}$$

and

$$e^{yx\frac{d}{dx}}f(x)=f(xe^y). \tag{3.8}$$

While (3.8) is straightforward to verify, (3.7) is surprisingly subtle (when logarithmic terms are involved). For instance, it turns out that (3.7) essentially amounts to a generating function form of the following combinatorial identity:

$$\frac{j!}{k!}\sum_{0<t_1<t_2<\cdots<t_{k-j}<k} t_1t_2\cdots t_{k-j} = \sum_{\substack{i_1+\cdots+i_j=k \\ 1\leq i_1,\ldots,i_j\leq k}} \frac{1}{i_1i_2\cdots i_j} \tag{3.9}$$

for all $k\in\mathbb{N}$ and $j=0,\ldots,k$. A wide variety of identities of this type will clearly arise as a result of the deeper part of our theory. Incidentally, at the time this work was being written, one of us (L.Z.) happened to pick up the then-current issue of the American Mathematical Monthly and happened to notice the following problem from the Problems and Solutions section, proposed by D. Lubell [Lu]:

> Let N and j be positive integers, and let $S=\{(w_1,\ldots,w_j)\in\mathbb{Z}_+^j\,|\,0<w_1+\cdots+w_j\leq N\}$ and $T=\{(w_1,\cdots,w_j)\in\mathbb{Z}_+^j\,|$ $w_1,\ldots,w_j$ are distinct and bounded by $N\}$. Show that
>
> $$\sum_S\frac{1}{w_1\cdots w_j}=\sum_T\frac{1}{w_1\cdots w_j}.$$

This follows immediately from (3.9), which is in fact a refinement, since the left-hand side of (3.9) is equal to

$$j!\sum_{1\leq w_1<w_2<\cdots<w_{j-1}\leq k-1}\frac{1}{w_1w_2\cdots w_{j-1}k}=\sum_{T_k}\frac{1}{w_1w_2\cdots w_j}$$

where $T_k=\{(w_1,\ldots,w_j)\in\{1,2,\ldots,k\}^j\,|\,w_i$ distinct, with maximum exactly $k\}$, the right-hand side is

$$\sum_{S_k}\frac{1}{w_1w_2\cdots w_j}$$

where $S_k=\{(w_1,\ldots,w_j)\in\{1,2,\ldots,k\}^j\,|\,w_1+\cdots+w_j=k\}$, and one has $S=\coprod_{k=1}^N S_k$ and $T=\coprod_{k=1}^N T_k$.

When we define the notion of logarithmic intertwining operator below, we will impose a condition requiring certain formal series to lie in spaces of the type $\mathcal{W}[\log x]\{x\}$ (so that for each power of x, possibly complex, we have a *polynomial* in $\log x$), partly because such results as the following (which is expected) will indeed hold in our formal setup when the powers of the formal variables are restricted in this way.

Lemma 3.6 *Let $a\in\mathbb{C}$ and $m\in\mathbb{Z}_+$. If $f(x)\in\mathcal{W}[\log x]\{x\}$ ($\mathcal{W}$ any vector space not involving x or $\log x$) satisfies the formal differential equation*

$$\left(x\frac{d}{dx}-a\right)^m f(x)=0, \tag{3.10}$$

then $f(x) \in \mathcal{W}x^a \oplus \mathcal{W}x^a \log x \cdots \oplus \mathcal{W}x^a (\log x)^{m-1}$*; and furthermore, if* m *is the smallest integer so that (3.10) is satisfied, then the coefficient of* $x^a(\log x)^{m-1}$ *in* $f(x)$ *is nonzero.*

This can be proved by induction on m.

Note that there are of course solutions of the equation (3.10) outside the space $\mathcal{W}[\log x]\{x\}$, for example, $f(x) = wx^b e^{(a-b)\log x} \in x^b\mathcal{W}[[\log x]]$ for any complex number $b \neq a$ and any nonzero vector $w \in \mathcal{W}$.

Following [M], with a slight generalization, we now introduce the notion of logarithmic intertwining operator. We will later see that the axioms in the following definition correspond exactly to those in the notion of certain "intertwining maps" (cf. Definition 4.1).

Definition 3.7 Let W_1, W_2, W_3 be generalized V-modules. A *logarithmic intertwining operator of type* $\binom{W_3}{W_1\,W_2}$ is a linear map

$$\mathcal{Y}(\cdot, x)\cdot : W_1 \otimes W_2 \to W_3[\log x]\{x\}, \tag{3.11}$$

or equivalently,

$$w_{(1)} \otimes w_{(2)} \mapsto \mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{n\in\mathbb{C}}\sum_{k\in\mathbb{N}} {w_{(1)}}^{\mathcal{Y}}_{n;\,k} w_{(2)} x^{-n-1}(\log x)^k \in W_3[\log x]\{x\} \tag{3.12}$$

for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, such that the following conditions are satisfied: the *lower truncation condition*: for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $n \in \mathbb{C}$,

$${w_{(1)}}^{\mathcal{Y}}_{n+m;\,k} w_{(2)} = 0 \quad \text{for} \quad m \in \mathbb{N} \text{ sufficiently large, independently of } k; \tag{3.13}$$

the *Jacobi identity*:

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) &Y(v, x_1)\mathcal{Y}(w_{(1)}, x_2)w_{(2)} \\ &- x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\mathcal{Y}(w_{(1)}, x_2)Y(v, x_1)w_{(2)} \\ = x_2^{-1}\delta&\left(\frac{x_1 - x_0}{x_2}\right)\mathcal{Y}(Y(v, x_0)w_{(1)}, x_2)w_{(2)} \end{aligned} \tag{3.14}$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (note that the first term on the left-hand side is meaningful because of (3.13)); and the $L(-1)$*-derivative property:* for any $w_{(1)} \in W_1$,

$$\mathcal{Y}(L(-1)w_{(1)}, x) = \frac{d}{dx}\mathcal{Y}(w_{(1)}, x). \tag{3.15}$$

Suppose in addition that V and W_1, W_2 and W_3 are strongly graded. A logarithmic intertwining operator $\mathcal{Y}$ is a *grading-compatible logarithmic intertwining operator* if for $\beta, \gamma \in \tilde{A}$ and $w_{(1)} \in W_1^{(\beta)}$, $w_{(2)} \in W_2^{(\gamma)}$, $n \in \mathbb{C}$ and $k \in \mathbb{N}$, we have

$${w_{(1)}}^{\mathcal{Y}}_{n;\,k} w_{(2)} \in W_3^{(\beta+\gamma)}. \tag{3.16}$$

By setting $v = \omega$ in (3.14) and then taking $\mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_1^{j+1}$ we have

$$[L(j), \mathcal{Y}(w_{(1)}, x)] = \sum_{i=0}^{j+1} \binom{j+1}{i} x^i \mathcal{Y}(L(j-i)w_{(1)}, x) \tag{3.17}$$

for any $w_{(1)} \in W_1$ and $j = -1, 0, 1$.

Every ordinary intertwining operator (as in, for example, [HL3]) is of course a logarithmic intertwining operator which involves no formal variable $\log x$. (The present lower truncation condition is more relaxed than the one in [HL3].) The grading-compatible logarithmic intertwining operators of a fixed type $\binom{W_3}{W_1\, W_2}$ form a vector space, which we denote by $\mathcal{V}^{W_3}_{W_1\, W_2}$. We call the dimension of $\mathcal{V}^{W_3}_{W_1\, W_2}$ the *fusion rule* for W_1, W_2 and W_3 and denote it by $N^{W_3}_{W_1\, W_2}$ (see Remark 3.11 below).

By direct analysis we can prove:

Lemma 3.8 *Let W_1, W_2, W_3 be generalized V-modules. Let*

$$\begin{aligned} \mathcal{Y}(\cdot, x)\cdot : W_1 \otimes W_2 &\to W_3\{x, \log x\} \\ w_{(1)} \otimes w_{(2)} &\mapsto \mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{n,k\in\mathbb{C}} {w_{(1)}}^{\mathcal{Y}}_{n;\,k} w_{(2)} x^{-n-1} (\log x)^k \end{aligned}$$

be a linear map that satisfies the L(-1)-derivative property (3.15) and the $L(0)$-bracket relation, that is, (3.17) with $j = 0$. Then for any $a, b, c \in \mathbb{C}$, $t \in \mathbb{N}$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$\begin{aligned} (L(0) - c)^t \mathcal{Y}(w_{(1)}, x)w_{(2)} = &\sum_{i,j,l\in\mathbb{N},\ i+j+l=t} \frac{t!}{i!j!l!} \cdot \\ &\cdot \left(x\frac{d}{dx} - c + a + b\right)^l \mathcal{Y}((L(0) - a)^i w_{(1)}, x)(L(0) - b)^j w_{(2)}. \end{aligned}$$

Also, for any $a, b, n, k \in \mathbb{C}$, $t \in \mathbb{N}$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, we have

$$\begin{aligned} &(L(0) - a - b + n + 1)^t ({w_{(1)}}^{\mathcal{Y}}_{n;\,k} w_{(2)}) \\ &\quad = t! \sum_{i,j,l\geq 0,\ i+j+l=t} \binom{k+l}{l} \left(\frac{(L(0) - a)^i}{i!} w_{(1)}\right)^{\mathcal{Y}}_{n;\,k+l} \left(\frac{(L(0) - b)^j}{j!} w_{(2)}\right); \end{aligned}$$

in generating function form, this gives

$$\begin{aligned} &e^{y(L(0)-a-b+n+1)} ({w_{(1)}}^{\mathcal{Y}}_{n;\,k} w_{(2)}) \\ &\quad = \sum_{l\in\mathbb{N}} \binom{k+l}{l} (e^{y(L(0)-a)} w_{(1)})^{\mathcal{Y}}_{n;\,k+l} (e^{y(L(0)-b)} w_{(2)}) y^l. \end{aligned} \tag{3.18}$$

Using this lemma, we have the following result, which summarizes some important properties of logarithmic intertwining operators:

Proposition 3.9 *Let W_1, W_2, W_3 be generalized V-modules, and let $\mathcal{Y}(\cdot, x)\cdot$ be a logarithmic intertwining operator of type $\binom{W_3}{W_1\, W_2}$. Let $w_{(1)}$ and $w_{(2)}$ be homogeneous elements of W_1 and W_2 of (generalized) weights n_1 and $n_2 \in \mathbb{C}$, respectively, and let k_1 and k_2 be positive integers such that $(L(0)-n_1)^{k_1} w_{(1)} = 0$ and $(L(0)-n_2)^{k_2} w_{(2)} = 0$. Then we have:*

(a) ([M]) For any $w'_{(3)} \in W_3^*$, $n_3 \in \mathbb{C}$ and $k_3 \in \mathbb{Z}_+$ such that $(L'(0) - n_3)^{k_3} w'_{(3)} = 0$,

$$\begin{aligned}&\langle w'_{(3)}, \mathcal{Y}(w_{(1)}, x) w_{(2)} \rangle\\ &\qquad \in \mathbb{C}x^{n_3-n_1-n_2} \oplus \mathbb{C}x^{n_3-n_1-n_2} \log x \oplus \cdots \oplus \mathbb{C}x^{n_3-n_1-n_2} (\log x)^{k_1+k_2+k_3-3}.\end{aligned}$$

(b) For any $n \in \mathbb{C}$ and $k \in \mathbb{N}$, $w_{(1)}{}^{\mathcal{Y}}_{n;\,k} w_{(2)} \in W_3$ is homogeneous of (generalized) weight $n_1 + n_2 - n - 1$.

(c) Fix $n \in \mathbb{C}$ and $k \in \mathbb{N}$. For each $i, j \in \mathbb{N}$, let m_{ij} be a nonnegative integer such that

$$(L(0) - n_1 - n_2 + n + 1)^{m_{ij}} (((L(0) - n_1)^i w_{(1)})^{\mathcal{Y}}_{n;\,k} (L(0) - n_2)^j w_{(2)}) = 0.$$

Then for all $t \geq \max\{m_{ij} \,|\, 0 \leq i < k_1,\ 0 \leq j < k_2\} + k_1 + k_2 - 2$,

$$w_{(1)}{}^{\mathcal{Y}}_{n;\,k+t} w_{(2)} = 0.$$

Proposition 3.9(b) immediately gives:

Corollary 3.10 *Let W_1, W_2 and W_3 be generalized V-modules whose weights are all congruent modulo $\mathbb{Z}$ to complex numbers h_1, h_2 and h_3, respectively (for example, when W_1, W_2 and W_3 are all indecomposable). Let $\mathcal{Y}(\cdot, x)\cdot$ be a logarithmic intertwining operator of type $\binom{W_3}{W_1\, W_2}$. Then all the powers of x in $\mathcal{Y}(\cdot, x)\cdot$ are congruent modulo $\mathbb{Z}$ to $h_3 - h_1 - h_2$.*

Remark 3.11 Let W_1, W_2 and W_3 be (ordinary) V-modules. Then it follows from Proposition 3.9(a), or alternatively, from Proposition 3.9(b) and (c), that any logarithmic intertwining operator of type $\binom{W_3}{W_1\, W_2}$ is just an ordinary intertwining operator of this type, i.e., it does not involve $\log x$. As a result, when V is a vertex operator algebra, for (ordinary) V-modules the notion of fusion rule defined in this paper coincides with the notion of fusion rule defined in, for example, [HL3] (except for the minor issue of the truncation condition.)

Our definition of logarithmic intertwining operator is identical to that in [M] (in case V is a vertex operator algebra) except that in [M], a logarithmic intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1\, W_2}$ is required to be a linear map $W_1 \to \mathrm{Hom}(W_2, W_3)\{x\}[\log x]$, instead of as in (3.11), and the lower truncation condition (3.13) is replaced by:

$$w_{(1)}{}^{\mathcal{Y}}_{n;k} w_{(2)} = 0 \ \text{ for } n \text{ whose real part is sufficiently large}$$

for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $k \in \mathbb{N}$.

Given a logarithmic intertwining operator $\mathcal{Y}$ as in (3.12), set

$$\mathcal{Y}^{(k)}(w_{(1)},x)w_{(2)} = \sum_{n\in\mathbb{C}} w_{(1)}{}^{\mathcal{Y}}_{n;\,k} w_{(2)} x^{-n-1}$$

for $k \in \mathbb{N}$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, so that

$$\mathcal{Y}(w_{(1)},x)w_{(2)} = \sum_{k\in\mathbb{N}} \mathcal{Y}^{(k)}(w_{(1)},x)w_{(2)}(\log x)^k.$$

Then each $\mathcal{Y}^{(k)}$ satisfies the Jacobi identity, but not in general the $L(-1)$-derivative property. However, the following result shows that suitable formal linear combinations of certain modifications of $\mathcal{Y}^{(k)}$ (depending on $t \in \mathbb{N}$; see below) form a sequence of logarithmic intertwining operators:

Proposition 3.12 *Let W_1, W_2, W_3 be generalized V-modules, and let $\mathcal{Y}(\cdot,x)\cdot$ be a logarithmic intertwining operator of type $\binom{W_3}{W_1\,W_2}$. For $\mu \in \mathbb{C}/\mathbb{Z}$ and $t \in \mathbb{N}$, define $\mathcal{X}_t^\mu : W_1 \otimes W_2 \to W_3[\log x]\{x\}$ by:*

$$\mathcal{X}_t^\mu : w_{(1)} \otimes w_{(2)} \mapsto \sum_{k\in\mathbb{N}} \binom{k+t}{t} \sum_{\bar{n}=\mu} w_{(1)}{}^{\mathcal{Y}}_{n;\,k+t} w_{(2)} x^{-n-1}(\log x)^k.$$

Then each $\mathcal{X}_t^\mu$ is a logarithmic intertwining operator of type $\binom{W_3}{W_1\,W_2}$. In particular, the operator $\mathcal{X}_t$ defined by

$$\mathcal{X}_t : w_{(1)} \otimes w_{(2)} \mapsto \sum_{k\in\mathbb{N}} \binom{k+t}{t} \sum_{n\in\mathbb{C}} w_{(1)}{}^{\mathcal{Y}}_{n;\,k+t} w_{(2)} x^{-n-1}(\log x)^k \tag{3.19}$$

is a logarithmic intertwining operator of the same type. In the strongly graded case, if $\mathcal{Y}$ is grading-compatible, then so are $\mathcal{X}_t^\mu$ and $\mathcal{X}_t$.

This can be proved essentially by induction on t.

Remark 3.13 Given any logarithmic intertwining operator $\mathcal{Y}(\cdot,x)\cdot$ as in (3.11) and any $i,j,k \in \mathbb{N}$, by Proposition 2.2 we see that

$$(L(0)-L(0)_s)^k \mathcal{Y}((L(0)-L(0)_s)^i\cdot,x)(L(0)-L(0)_s)^j\cdot$$

is again a logarithmic intertwining operator, and in the strongly graded case, if $\mathcal{Y}$ is grading-compatible, so is this operator. It turns out that the intertwining operators (3.19) are just linear combinations of these.

Now we define operators "$x^{\pm L(0)}$" for generalized modules, in the natural way:

Definition 3.14 Let W be a generalized V-module. We define $x^{\pm L(0)} : W \to W\{x\}[\log x] \subset W[\log x]\{x\}$ as follows: For any $w \in W_{[n]}$ ($n \in \mathbb{C}$), define

$$x^{\pm L(0)}w = x^{\pm n} e^{\pm \log x (L(0)-n)} w$$

(note that the local nilpotence of $L(0)-n$ on $W_{[n]}$ insures that the formal exponential series terminates) and then extend linearly to all $w \in W$. We also define operators $x^{\pm L'(0)}$ on W^* by the condition that for all $w' \in W^*$ and $w \in W$,

$$\langle x^{\pm L'(0)}w', w\rangle = \langle w', x^{\pm L(0)}w\rangle \; (\in \mathbb{C}\{x\}[\log x]),$$

so that $x^{\pm L'(0)} : W^* \to W^*\{x\}[[\log x]]$.

Remark 3.15 Note that for $w \in W_{[n]}$, by definition we have

$$x^{\pm L(0)}w = x^{\pm n}\sum_{i\in\mathbb{N}} \frac{(L(0)-n)^i w}{i!}(\pm \log x)^i \in x^{\pm n}W_{[n]}[\log x]. \tag{3.20}$$

It is also easy to verify that for any $w \in W$,

$$x^{L(0)}x^{-L(0)}w = w = x^{-L(0)}x^{L(0)}w, \tag{3.21}$$

$$\frac{d}{dx}x^{\pm L(0)}w = \pm x^{-1}x^{\pm L(0)}L(0)w. \tag{3.22}$$

We have the following generalizations to logarithmic intertwining operators of three standard formulas for (ordinary) intertwining operators (see [FHL], formulas (5.4.21), (5.4.22) and (5.4.23)); see also [M] for parts (a) and (b):

Proposition 3.16 *Let $\mathcal{Y}$ be a logarithmic intertwining operator of type $\binom{W_3}{W_1\, W_2}$ and let $w \in W_1$. Then*

(a)

$$e^{yL(-1)}\mathcal{Y}(w,x)e^{-yL(-1)} = \mathcal{Y}(e^{yL(-1)}w,x) = \mathcal{Y}(w,x+y)$$

(recall (3.3))

(b)

$$y^{L(0)}\mathcal{Y}(w,x)y^{-L(0)} = \mathcal{Y}(y^{L(0)}w,xy)$$

(recall (3.6))

(c)

$$e^{yL(1)}\mathcal{Y}(w,x)e^{-yL(1)} = \mathcal{Y}(e^{y(1-yx)L(1)}(1-yx)^{-2L(0)}w, x(1-yx)^{-1}).$$

The equality (a) essentially follows from the identity

$$L(-1)\mathcal{Y}(w,x) = \mathcal{Y}(L(-1)w,x) + \mathcal{Y}(w,x)L(-1), \quad w \in W,$$

and (3.7) in Proposition 3.5, (b) essentially follows from (3.6), (3.18) and (3.21), and the proof of (c) uses results in [FHL].

Let $\mathcal{Y}$ be a logarithmic intertwining operator of type $\binom{W_3}{W_1\, W_2}$. By analogy with the ordinary case in Section 7.1 of [HL4], for any complex number ζ, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$,

$$\mathcal{Y}(w_{(1)},y)w_{(2)}\Big|_{y^n=e^{n\zeta}x^n,\ (\log y)^k=(\zeta+\log x)^k,\ n\in\mathbb{C},\ k\in\mathbb{N}} \tag{3.23}$$

is a well-defined element of $W_3[\log x]\{x\}$. We denote this element by $\mathcal{Y}(w_{(1)}, e^{\zeta}x)w_{(2)}$. Note that it depends on ζ, not just on e^{ζ}. Given any $r \in \mathbb{Z}$, we define $\Omega_r(\mathcal{Y}) : W_2 \otimes W_1 \to W_3[\log x]\{x\}$ by

$$\Omega_r(\mathcal{Y})(w_{(2)}, x)w_{(1)} = e^{xL(-1)}\mathcal{Y}(w_{(1)}, e^{(2r+1)\pi i}x)w_{(2)} \tag{3.24}$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, the same formula as formula (7.1) of [HL4]. We can prove that Proposition 7.1 in [HL4] still holds when we replace the phrase "an intertwining operator $\mathcal{Y}$" by "a logarithmic intertwining operator $\mathcal{Y}$." This result continues to hold for grading-compatible logarithmic intertwining operators, in the strongly graded case, and in particular, $\mathcal{Y} \mapsto \Omega_r(\mathcal{Y})$ is a linear isomorphism from $\mathcal{V}^{W_3}_{W_1\,W_2}$ to $\mathcal{V}^{W_3}_{W_2\,W_1}$ with inverse given by Ω_{-r-1}.

In case V, W_1, W_2 and W_3 are strongly graded, we can also carry over the concept of "r-contragredient operator" from [HL4], as follows: Given a grading-compatible logarithmic intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1\,W_2}$ and an integer r, we define the *r-contragredient operator of $\mathcal{Y}$* to be the linear map

$$\begin{array}{rcl} W_1 \otimes W_3' & \to & W_2'\{x\}[[\log x]] \\ w_{(1)} \otimes w'_{(3)} & \mapsto & A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)} \end{array}$$

given by

$$\begin{aligned} &\langle A_r(\mathcal{Y})(w_{(1)}, x)w'_{(3)}, w_{(2)}\rangle_{W_2} \\ &\quad = \langle w'_{(3)}, \mathcal{Y}(e^{xL(1)}e^{(2r+1)\pi i L(0)}(x^{-L(0)})^2 w_{(1)}, x^{-1})w_{(2)}\rangle_{W_3}, \end{aligned}$$

for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W_3'$, where we use the notation

$$f(x^{-1}) = \sum_{m\in\mathbb{N},\, n\in\mathbb{C}} w_{n,m}x^{-n}(-\log x)^m$$

for any $f(x) = \sum_{m\in\mathbb{N},\, n\in\mathbb{C}} w_{n,m}x^n(\log x)^m \in \mathcal{W}\{x\}[[\log x]]$, $\mathcal{W}$ any vector space (not involving x). Note that for the case $W_1 = V$, $W_2 = W_3 = W$ and $\mathcal{Y} = Y_W$, the operator $A_r(Y_W)$ agrees with the contragredient vertex operator Y_W' of Y_W (recall (2.3) and (2.4)) for any $r \in \mathbb{Z}$. In general, we can prove that Proposition 7.3 in [HL4] still holds when we replace the phrase "an intertwining operator $\mathcal{Y}$" by "a grading-compatible logarithmic intertwining operator $\mathcal{Y}$," and in particular $\mathcal{Y} \mapsto A_r(\mathcal{Y})$ defines a linear isomorphism from $\mathcal{V}^{W_3}_{W_1\,W_2}$ to $\mathcal{V}^{W_2'}_{W_1\,W_3'}$, with inverse given by A_{-r-1}.

Let V, W_1, W_2 and W_3 be strongly graded. Set

$$N_{W_1W_2W_3} = N^{W_3'}_{W_1\,W_2}.$$

Then from the above discussion, we see that, as in the ordinary case, for any permutation σ of $(1, 2, 3)$, $N_{W_1W_2W_3} = N_{W_{\sigma(1)}W_{\sigma(2)}W_{\sigma(3)}}$.

Finally, it is clear from Proposition 3.9(b) that, in the nontrivial logarithmic intertwining operator case, taking projections to (generalized) weight subspaces is not enough to recover the coefficients of $x^n(\log x)^k$ in $\mathcal{Y}(w_{(1)}, x)w_{(2)}$ for each $n \in \mathbb{C}$ and $k \in \mathbb{N}$. However, by using identities related to those in Lemma 3.8 we have the following result:

Proposition 3.17 *Let W_1, W_2, W_3 be generalized V-modules and $\mathcal{Y}$ a logarithmic intertwining operator of type $\binom{W_3}{W_1\,W_2}$. Let $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ be homogeneous of (generalized) weights n_1 and n_2, respectively. Then for any $n \in \mathbb{C}$ and any $r \in \mathbb{N}$, $w_{(1)}{}^{\mathcal{Y}}_{n;\,r}w_{(2)}$ can be written as a certain linear combination of products of the component of weight $n_1 + n_2 - n - 1$ of*

$$(L(0) - n_1 - n_2 + n + 1)^l \mathcal{Y}((L(0) - n_1)^i w_{(1)}, x)(L(0) - n_2)^j w_{(2)}$$

for certain $i, j, l \in \mathbb{N}$ with monomials of the form $x^{n+1}(\log x)^m$ for certain $m \in \mathbb{N}$.

4 Definition and constructions of $P(z)$-tensor product

We now present our results on generalization of the $P(z)$-tensor product construction developed in [HL3], [HL4] and [HL5]. Although many results here have the same statement as in the ordinary case, one should note that the involvement of logarithms makes the situation subtle, and their validity is based on the results from the last section.

Throughout this section and the remainder of this paper, we shall assume the following, unless other assumptions are explicitly made: A is an abelian group and $\tilde{A}$ is an abelian group containing A as a subgroup; V is a strongly A-graded conformal vertex algebra; all V-modules and generalized V-modules considered are strongly $\tilde{A}$-graded; and all intertwining operators and logarithmic intertwining operators considered are grading-compatible. Also, z will be a fixed nonzero complex number.

We first generalize the notion of $P(z)$-intertwining map from Section 4 of [HL3]:

Definition 4.1 Let (W_1, Y_1), (W_2, Y_2) and (W_3, Y_3) be generalized V-modules. A *$P(z)$-intertwining map of type $\binom{W_3}{W_1\,W_2}$* is a linear map $I : W_1 \otimes W_2 \to \overline{W}_3$ such that the following conditions are satisfied: the *grading compatibility condition*: for $\beta, \gamma \in \tilde{A}$ and $w_{(1)} \in W_1^{(\beta)}$, $w_{(2)} \in W_2^{(\gamma)}$,

$$I(w_{(1)} \otimes w_{(2)}) \in \overline{W_3^{(\beta+\gamma)}}; \tag{4.1}$$

the *lower truncation condition:* for any elements $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, and any $n \in \mathbb{C}$,

$$\pi_{n-m} I(w_{(1)} \otimes w_{(2)}) = 0 \quad \text{for} \quad m \in \mathbb{N} \ \text{ sufficiently large} \tag{4.2}$$

(which in fact follows from (4.1)); and the *Jacobi identity*:

$$x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right) Y_3(v, x_1) I(w_{(1)} \otimes w_{(2)})$$

$$= z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) I(Y_1(v, x_0)w_{(1)} \otimes w_{(2)})$$
$$+ x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right) I(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \tag{4.3}$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (note that the left-hand side of (4.3) is meaningful because any infinite linear combination of v_n ($n \in \mathbb{Z}$) of the form $\sum_{n<N} a_n v_n$ ($a_n \in \mathbb{C}$) acts on any $I(w_{(1)} \otimes w_{(2)})$, due to (4.2)). The vector space of $P(z)$-intertwining maps of type $\binom{W_3}{W_1 W_2}$ is denoted by $\mathcal{M}[P(z)]_{W_1 W_2}^{W_3}$, or simply by $\mathcal{M}_{W_1 W_2}^{W_3}$ if there is no ambiguity.

Following [HL3] we will choose the branch of $\log z$ so that $0 \leq \mathrm{Im}(\log z) < 2\pi$. We will also use the notation $l_p(z) = \log z + 2\pi i p$, $p \in \mathbb{Z}$, from [HL3] for arbitrary branches of the log function. For any expression $f(x)$ as in (3.1) and any $\zeta \in \mathbb{C}$, whenever

$$f(x)\Big|_{x^n = e^{n\zeta},\ (\log x)^m = \zeta^m,\ m, n \in \mathbb{C}} \tag{4.4}$$

exists, in particular, when $f(x) = \mathcal{Y}(w_{(1)}, x)w_{(2)}$ ($\in W_3[\log x]\{x\}$) for some $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and some logarithmic intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1\, W_2}$ ((4.4) exists in this case because of Proposition 3.9(b)), we will write (4.4) simply as $f(x)\Big|_{x=e^{\zeta}}$ or $f(e^{\zeta})$, and call this "substituting e^{ζ} for x in $f(x)$", despite the fact that in general it depends on ζ, not just e^{ζ} (see also (3.23)). In addition, fixing an integer p, we will sometimes write

$$f(x)\Big|_{x=z} \quad \text{or} \quad f(z)$$

instead of $f(x)\Big|_{x=e^{l_p(z)}}$ or $f(e^{l_p(z)})$.

Fix an integer p. Let $\mathcal{Y}$ be a logarithmic intertwining operator of type $\binom{W_3}{W_1\, W_2}$. Then we have a linear map $I_{\mathcal{Y},p} : W_1 \otimes W_2 \to \overline{W}_3$ defined by

$$I_{\mathcal{Y},p}(w_{(1)} \otimes w_{(2)}) = \mathcal{Y}(w_{(1)}, e^{l_p(z)})w_{(2)} \tag{4.5}$$

for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (the right-hand side of (4.5) is indeed an element of $\overline{W}_3$, again because of Proposition 3.9(b)). On the other hand, given a $P(z)$-intertwining map I, we define a linear map $\mathcal{Y}_{I,p} : W_1 \otimes W_2 \to W_3[\log x]\{x\}$ by

$$\mathcal{Y}_{I,p}(w_{(1)}, x)w_{(2)} =$$
$$y^{L(0)} x^{L(0)} I(y^{-L(0)} x^{-L(0)} w_{(1)} \otimes y^{-L(0)} x^{-L(0)} w_{(2)})\Big|_{y = e^{-l_p(z)}} \tag{4.6}$$

for any $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (this is well defined and indeed maps to $W_3[\log x]\{x\}$, in view of (3.20) and (4.2)). Define the notation $w_{(1)}{}^{I,p}_{n;k} w_{(2)} \in W_3$

by

$$\mathcal{Y}_{I,p}(w_{(1)},x)w_{(2)} = \sum_{n\in\mathbb{C}}\sum_{k\in\mathbb{N}} w_{(1)}{}^{I,p}_{n;k}w_{(2)}x^{-n-1}(\log x)^k.$$

Observe that since the operator $x^{\pm L(0)}$ always increases the power of x in an expression homogeneous of (generalized) weight n by $\pm n$, from (4.6) we have that $w_{(1)}{}^{I,p}_{n;k}w_{(2)} \in (W_3)_{[n_1+n_2-n-1]}$ for $w_{(1)} \in (W_1)_{[n_1]}$ and $w_{(2)} \in (W_2)_{[n_2]}$.

Using results in the last section we have the following correspondence between logarithmic intertwining operators and $P(z)$-intertwining maps, generalizing Proposition 12.2 in [HL5]:

Proposition 4.2 *For $p \in \mathbb{Z}$, the correspondence $\mathcal{Y} \mapsto I_{\mathcal{Y},p}$ is a linear isomorphism from the space $\mathcal{V}^{W_3}_{W_1W_2}$ of logarithmic intertwining operators of type $\binom{W_3}{W_1\,W_2}$ to the space $\mathcal{M}^{W_3}_{W_1W_2}$ of $P(z)$-intertwining maps of the same type. Its inverse map is given by $I \mapsto \mathcal{Y}_{I,p}$.*

The definition of $P(z)$-tensor product, in terms of the $P(z)$-intertwining maps defined earlier, is the same as that in [HL3], except that here we take into account that the module category under consideration can vary. For this, recall the category $\mathcal{C}_1$ from the end of Section 2. We have:

Definition 4.3 For $W_1, W_2 \in \mathrm{ob}\,\mathcal{C}_1$, a *$P(z)$-product of W_1 and W_2* is an object (W_3, Y_3) in $\mathcal{C}_1$ together with a $P(z)$-intertwining map I_3 of type $\binom{W_3}{W_1\,W_2}$. We denote it by $(W_3, Y_3; I_3)$ or simply by (W_3, I_3). Let $(W_4, Y_4; I_4)$ be another $P(z)$-product of W_1 and W_2. A *morphism* from $(W_3, Y_3; I_3)$ to $(W_4, Y_4; I_4)$ is a module map η from W_3 to W_4 such that $I_4 = \bar{\eta} \circ I_3$, where $\bar{\eta}$ is the natural map from $\overline{W}_3$ to $\overline{W}_4$ uniquely extending η.

Definition 4.4 Let $\mathcal{C}$ be a full subcategory of $\mathcal{C}_1$. For $W_1, W_2 \in \mathrm{ob}\,\mathcal{C}$, a *$P(z)$-tensor product of W_1 and W_2 in $\mathcal{C}$* is a $P(z)$-product $(W_0, Y_0; I_0)$ with $W_0 \in \mathrm{ob}\,\mathcal{C}$ such that for any $P(z)$-product $(W, Y; I)$ with $W \in \mathrm{ob}\,\mathcal{C}$, there is a unique morphism from $(W_0, Y_0; I_0)$ to $(W, Y; I)$. Clearly, a $P(z)$-tensor product of W_1 and W_2 in $\mathcal{C}$, if it exists, is unique up to a unique isomorphism. In this case we will denote it as $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ and call the object $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})$ the *$P(z)$-tensor product module of W_1 and W_2 in $\mathcal{C}$*. We will omit the phrase "in $\mathcal{C}$" if the category $\mathcal{C}$ under consideration is understood in context.

Remark 4.5 It is easy to show that in this setting, if $W_1 \boxtimes_{P(z)} W_2$ exists, then $W_1 \boxtimes_{P(z_1)} W_2$ exists *for any $z_1 \in \mathbb{C}^\times$*.

Proposition 4.2 shows that Proposition 12.3 in [HL5], which relates module maps from a $P(z)$-tensor product module with $P(z)$-intertwining maps and intertwining operators, still holds, for logarithmic intertwining operators in the present case.

Let $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$ be the $P(z)$-tensor product of W_1 and W_2. We will sometimes denote the image of the $P(z)$-intertwining map

$$w_{(1)} \otimes w_{(2)} \mapsto \boxtimes_{P(z)}(w_{(1)} \otimes w_{(2)}) = \boxtimes_{P(z)}(w_{(1)}, z)w_{(2)}$$

simply by $w_{(1)} \boxtimes_{P(z)} w_{(2)}$.

Remark 4.6 Suppose that $\mathcal{C}$ is a full subcategory of $\mathcal{C}_1$ such that for all $W_1, W_2 \in \mathrm{ob}\,\mathcal{C}$, the $P(z)$-tensor product of W_1 and W_2 exists in $\mathcal{C}$. Then the $P(z)$-tensor product can be viewed as a (bi)functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$ in a natural way.

As in the ordinary case, we have:

Proposition 4.7 *The module $W_1 \boxtimes_{P(z)} W_2$ (if it exists) is spanned (as a vector space) by the (generalized-)weight components of the elements of $\overline{W_1 \boxtimes_{P(z)} W_2}$ of the form $w_{(1)} \boxtimes_{P(z)} w_{(2)}$, for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$.*

Let $\mathcal{C}$ be some full subcategory of $\mathcal{C}_1$ closed under the contragredient functor $(\cdot)'$ and under taking finite direct sums, submodules and quotients, and containing V as an object.

We now use the "double dual" approach, as in [HL3], [HL4] and [HL5], to construct the $P(z)$-tensor product in $\mathcal{C}$, when it exists. First we study a certain adjoint of a $P(z)$-intertwining map. Recall from [FLM] that ι_+ is the map that expands a formal rational function (in t, below) in the direction of nonnegative powers of the formal variable. Also recall from [HL3] the notation

$$Y_t(v, x) = \sum_{n \in \mathbb{Z}} (v \otimes t^n) x^{-n-1}$$

for $v \in V$.

Definition 4.8 Define the linear action $\tau_{P(z)}$ of

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$$

on $(W_1 \otimes W_2)^*$ by (cf. (5.1) in [HL3])

$$\begin{aligned}\left(\tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1^{-1} - z}{x_0}\right)Y_t(v, x_1)\right)\lambda\right)&(w_{(1)} \otimes w_{(2)})\\ = z^{-1}\delta\left(\frac{x_1^{-1} - x_0}{z}\right)&\lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)})\\ +x_0^{-1}\delta\left(\frac{z - x_1^{-1}}{-x_0}\right)&\lambda(w_{(1)} \otimes Y_2^o(v, x_1)w_{(2)})\end{aligned}$$

for $v \in V$, $\lambda \in (W_1 \otimes W_2)^*$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$. (Note that this is indeed well defined and the right-hand side is in fact a finite sum in view of the lower truncation condition for vertex operators.) Denote by $Y'_{P(z)}$ the action of $V \otimes \mathbb{C}[t, t^{-1}]$ on $(W_1 \otimes W_2)^*$ thus defined: $Y'_{P(z)}(v, x) = \tau_{P(z)}(Y_t(v, x))$, that is,

$$\begin{aligned}(Y'_{P(z)}(v, x_1)\lambda)(w_{(1)} \otimes w_{(2)}) &= \lambda(w_{(1)} \otimes Y_2^o(v, x_1)w_{(2)})\\ &+\mathrm{Res}_{x_0} z^{-1}\delta\left(\frac{x_1^{-1} - x_0}{z}\right)\lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2})^{L(0)}v, x_0)w_{(1)} \otimes w_{(2)}).\end{aligned}$$

Remark 4.9 Using the action $\tau_{P(z)}$, (4.3) can be equivalently written as

$$\left(x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right)Y'(v,x_1)w'_{(3)}\right)\circ I=\tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right)Y_t(v,x_1)\right)(w'_{(3)}\circ I).$$

for $w'_{(3)}\in W'_3$.

Using $\tau_{P(z)}$ we find that Proposition 13.3 in [HL5] and the corresponding commutator formula still hold.

Write

$$Y'_{P(z)}(\omega,x)=\sum_{n\in\mathbb{Z}}L'_{P(z)}(n)x^{-n-2}.$$

Then we have that the coefficient operators of $Y'_{P(z)}(\omega,x)$ satisfy the Virasoro algebra commutator relations, that is,

$$[L'_{P(z)}(m),L'_{P(z)}(n)]=(m-n)L'_{P(z)}(m+n)+\frac{1}{12}(m^3-m)\delta_{m+n,0}c.$$

Another notion, corresponding to the lower truncation condition for $P(z)$-intertwining maps, is needed:

Definition 4.10 A map $J\in\mathrm{Hom}(W'_3,(W_1\otimes W_2)^*)$ is said to be *grading restricted* if J respects the $\tilde{A}$-gradings (the notion of homogeneity of an element of $(W_1\otimes W_2)^*$ with respect to $\tilde{A}$ is defined in the obvious way using the usual tensor product $\tilde{A}$-grading of $W_1\otimes W_2$) and if given any $n\in\mathbb{C}$, $w_{(1)}\in W_1$ and $w_{(2)}\in W_2$,

$$J((W'_3)_{[n-m]})(w_{(1)}\otimes w_{(2)})=0\quad\text{for}\quad m\in\mathbb{N}\ \text{ sufficiently large.}$$

Clearly, $J\in\mathrm{Hom}(W'_3,(W_1\otimes W_2)^*)$ is grading restricted if and only if the map $I\in\mathrm{Hom}(W_1\otimes W_2,\overline{W}_3)$ defined by

$$\langle w'_{(3)},I(w_{(1)}\otimes w_{(2)})\rangle=\langle J(w'_{(3)}),w_{(1)}\otimes w_{(2)}\rangle$$

satisfies the grading-compatibility condition (4.1) and the lower truncation condition (4.2).

From this and Remark 4.9 we have the following result generalizing Proposition 13.1 in [HL5]:

Proposition 4.11 *Let W_1, W_2 and W_3 be objects in $\mathcal{C}_1$. Then under the natural map*

$$\mathrm{Hom}(W_1\otimes W_2,\overline{W}_3)\to\mathrm{Hom}(W'_3,(W_1\otimes W_2)^*),\tag{4.7}$$

the $P(z)$-intertwining maps of type $\binom{W_3}{W_1\,W_2}$ correspond exactly to the grading restricted maps in $\mathrm{Hom}(W'_3,(W_1\otimes W_2)^*)$ intertwining the actions of

$$V\otimes\iota_+\mathbb{C}[t,t^{-1},(z^{-1}-t)^{-1}]$$

on W'_3 and $(W_1\otimes W_2)^*$.

Remark 4.12 Note that in contrast with the case in [HL5], the natural map (4.7) is in general only an injection, rather than an isomorphism. This is because $\overline{W}_3$ is the completion of W_3 with respect to the $\mathbb{C}$-grading and not the $\tilde{A}$-grading.

For $W_1, W_2 \in \mathrm{ob}\,\mathcal{C}_1$, a $P(z)$-product (W, I) of W_1 and W_2 with $W \in \mathrm{ob}\,\mathcal{C}_1$ can now be reformulated as an object (W, Y) in $\mathcal{C}_1$ equipped with a linear map I from $W_1 \otimes W_2$ to $\overline{W}$ such that the corresponding linear map

$$\begin{aligned} I' : W' &\to (W_1 \otimes W_2)^*, \\ w' &\mapsto w' \circ I \end{aligned}$$

is grading restricted and satisfies the intertwining conditions in Proposition 4.11.

Let $W_1, W_2 \in \mathrm{ob}\,\mathcal{C}$. Define $W_1 \boxslash_{P(z)} W_2$ to be the sum (or union) of all $I'(W') \subset (W_1 \otimes W_2)^*$, where (W, I) is a $P(z)$-product of W_1 and W_2 with $W \in \mathrm{ob}\,\mathcal{C}$. Then from the constructions and a proof analogous to that in the ordinary case we have the following generalization of Proposition 13.7 in [HL5]:

Proposition 4.13 *If $(W_1 \boxslash_{P(z)} W_2, Y'_{P(z)})$ is an object of $\mathcal{C}$, then denoting by $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})$ its contragredient module, we have that the $P(z)$-tensor product in $\mathcal{C}$ exists and is $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; i')$, where i is the natural inclusion from $W_1 \boxslash_{P(z)} W_2$ to $(W_1 \otimes W_2)^*$. Conversely, if the $P(z)$-tensor product of W_1 and W_2 in $\mathcal{C}$ exists, then $(W_1 \boxslash_{P(z)} W_2, Y'_{P(z)})$ is an object in $\mathcal{C}$.*

Let (W, I) be a $P(z)$-product of W_1 and W_2 and $w' \in W'$. It is easy to see that $I'(w')$ satisfies the following nontrivial and subtle conditions on $\lambda \in (W_1 \otimes W_2)^*$:

The $P(z)$-compatibility condition

(a) The *lower truncation condition*: For all $v \in V$, the formal Laurent series $Y'_{P(z)}(v, x)\lambda$ involves only finitely many negative powers of x.

(b) The following formula holds:

$$\begin{aligned} &\tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1^{-1} - z}{x_0}\right)Y_t(v, x_1)\right)\lambda \\ &\quad = x_0^{-1}\delta\left(\frac{x_1^{-1} - z}{x_0}\right)Y'_{P(z)}(v, x_1)\lambda \quad \text{for all } \ v \in V. \end{aligned} \tag{4.8}$$

(Note that the two sides of (4.8) are not *a priori* equal for general $\lambda \in (W_1 \otimes W_2)^*$.)

Furthermore, since I' in particular intertwines the two actions of $V \otimes \mathbb{C}[t, t^{-1}]$, $I'(W')$ is a generalized V-module. Therefore, for each $w' \in W'$, $I'(w')$ also satisfies the following condition on an element $\lambda \in (W_1 \otimes W_2)^*$:

The $P(z)$-local grading restriction condition

(a) The *grading condition*: λ is a (finite) sum of generalized eigenvectors in $(W_1 \otimes W_2)^*$ for the operator $L'_{P(z)}(0)$ that are also homogeneous with respect to $\tilde{A}$.

(b) The smallest $\tilde{A}$-graded subspace W_λ of $(W_1 \otimes W_2)^*$ containing λ and stable under the component operators $\tau_{P(z)}(v \otimes t^m)$ of the operators $Y'_{P(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$ has the properties

$$\dim(W_\lambda)^{(\beta)}_{[n]} < \infty,$$
$$(W_\lambda)^{(\beta)}_{[n+k]} = 0 \quad \text{for} \quad k \in \mathbb{Z} \quad \text{sufficiently negative}$$

for any $n \in \mathbb{C}$ and $\beta \in \tilde{A}$, where the subscripts denote the $\mathbb{C}$-grading by $L'_{P(z)}(0)$-eigenvalues and the superscripts denote the natural $\tilde{A}$-grading.

The next two results are crucial to the theory, as the corresponding results were in [HL3]–[HL5]. Using the results on logarithmic operators in the last section we are able to prove, by methods analogous to, but more subtle than, the methods in [HL3]–[HL5]:

Theorem 4.14 *Let λ be an element of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$-compatibility condition. Then when acting on λ, the Jacobi identity for $Y'_{P(z)}$ holds, that is,*

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y'_{P(z)}(u, x_1) Y'_{P(z)}(v, x_2)\lambda \\
&\quad - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y'_{P(z)}(v, x_2) Y'_{P(z)}(u, x_1)\lambda \\
&= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y'_{P(z)}(Y(u, x_0)v, x_2)\lambda
\end{aligned}$$

for $u, v \in V$.

Proposition 4.15 *The subspace consisting of the elements of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$-compatibility condition is stable under the operators $\tau_{P(z)}(v \otimes t^n)$ for $v \in V$ and $n \in \mathbb{Z}$, and similarly for the subspace consisting of the elements satisfying the $P(z)$-local grading-restriction condition.*

As a consequence of these results, we have that an element of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$-compatibility condition and the $P(z)$-local grading restriction condition generates a (strongly graded) generalized V-module under the operators $\tau_{P(z)}(v \otimes t^n)$ for $v \in V$ and $n \in \mathbb{Z}$.

From all these results we can use the strategy of [HL3]–[HL5] to obtain the following alternative description of $W_1 \boxtimes_{P(z)} W_2$:

Proposition 4.16 *Suppose that for every element λ of $(W_1 \otimes W_2)^*$ satisfying the $P(z)$-compatibility condition and the $P(z)$-local grading restriction condition, the generalized module W_λ generated by λ under the operators $\tau_{P(z)}(v \otimes t^n)$ for $v \in V$ and $n \in \mathbb{Z}$ lies in $\mathcal{C}$ (this of course holds in particular if $\mathcal{C} = \mathcal{C}_1$). Then the subspace of $(W_1 \otimes W_2)^*$ consisting of all such elements is equal to $W_1 \boxtimes_{P(z)} W_2$.*

In the earlier work [HL3]–[HL5] of the first two authors, another type of tensor product, the $Q(z)$-tensor product, was studied and developed in addition to the $P(z)$-tensor product, in [HL3] and [HL4]. Then the $P(z)$-tensor product was studied systematically in [HL5], and many proofs for the $P(z)$ case were obtained by the use of the results established for the $Q(z)$ case. Below we give the definition of a $Q(z)$-intertwining map and its relations to $P(z)$-intertwining maps, in the present more general context.

Definition 4.17 Let (W_1, Y_1), (W_2, Y_2) and (W_3, Y_3) be generalized V-modules. A $Q(z)$*-intertwining map of type* $\binom{W_3}{W_1\,W_2}$ is a linear map $I : W_1 \otimes W_2 \to \overline{W}_3$ such that the following conditions are satisfied: the *grading compatibility condition*: for $\beta, \gamma \in \tilde{A}$ and $w_{(1)} \in W_1^{(\beta)}$, $w_{(2)} \in W_2^{(\gamma)}$,

$$I(w_{(1)} \otimes w_{(2)}) \in \overline{W_3^{(\beta+\gamma)}}; \tag{4.9}$$

the *lower truncation condition:* for any elements $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, and any $n \in \mathbb{C}$,

$$\pi_{n-m} I(w_{(1)} \otimes w_{(2)}) = 0 \quad \text{for} \quad m \in \mathbb{N} \ \text{ sufficiently large}; \tag{4.10}$$

and the *Jacobi identity*:

$$\begin{aligned} z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) & Y_3^o(v, x_0) I(w_{(1)} \otimes w_{(2)}) \\ &= x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right) I(Y_1^o(v, x_1)w_{(1)} \otimes w_{(2)}) \\ &\quad - x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right) I(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \end{aligned} \tag{4.11}$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (recall (2.3), and note that the left-hand side of (4.11) is meaningful because any infinite linear combination of v_n of the form $\sum_{n<N} a_n v_n$ ($a_n \in \mathbb{C}$) acts on any $I(w_{(1)} \otimes w_{(2)})$, due to (4.10)). The vector space of $Q(z)$-intertwining maps of type $\binom{W_3}{W_1\,W_2}$ is denoted by $\mathcal{M}[Q(z)]^{W_3}_{W_1 W_2}$, or simply by $\mathcal{M}^{W_3}_{W_1 W_2}$ if there is no ambiguity.

We have the following result relating $P(z)$- and $Q(z)$-intertwining maps, essentially due to the fact that $Q(z)$ represents the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with ordered punctures z, ∞, 0, *with z the negatively oriented puncture*, and with standard local coordinates vanishing at these punctures, while $P(z)$ represents the Riemann sphere with ordered punctures ∞, z, 0, *with ∞ the negatively oriented puncture*, and with standard local coordinates vanishing at the punctures:

Proposition 4.18 *Let W_1, W_2 and W_3 be generalized V-modules. Let $I : W_1 \otimes W_2 \to \overline{W}_3$ and $J : W_3' \otimes W_2 \to \overline{W_1'}$ be linear maps related to each other by*

$$\langle w_{(1)}, J(w'_{(3)} \otimes w_{(2)}) \rangle = \langle w'_{(3)}, I(w_{(1)} \otimes w_{(2)}) \rangle$$

for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w'_{(3)} \in W_3'$. Then I is a $Q(z)$-intertwining map of type $\binom{W_3}{W_1\,W_2}$ if and only if J is a $P(z)$-intertwining map of type $\binom{W_1'}{W_3'\,W_2}$.

We remark that all the material in the earlier part of this section can be carried over to the $Q(z)$ case.

5 The associativity isomorphism

Having developed the concept of logarithmic intertwining operator and constructed the $P(z)$-tensor product, when it exists, we now proceed to the construction of the associativity isomorphism for this type of tensor product. We are able to show that the material in [H1] can be carried over, or adapted, to the setup in this paper. We first investigate the necessary conditions for the existence of the associativity isomorphism, and then we give the construction under these conditions. Again we refer the reader to [HLZ] for further details and proofs.

We adopt the definition (incorporating convergence) of existence of products (respectively, iterates) of two intertwining maps given in (14.1) (respectively, (14.2)) in [H1]. We have the following result generalizing that in [H1], which can be proved by a logarithmic analogue of the arguments in [H1]. One needs to use the transformations Ω_r developed in Section 3, which involve $\log x$ in a subtle way.

Proposition 5.1 *The following two conditions are equivalent:*

1. *For any objects W_1, W_2, W_3, W_4 and M_1 in $\mathcal{C}$, any nonzero complex numbers z_1 and z_2 satisfying $|z_1| > |z_2| > 0$, any $P(z_1)$-intertwining map I_1 of type $\binom{W_4}{W_1 M_1}$ and any $P(z_2)$-intertwining map I_2 of type $\binom{M_1}{W_2 W_3}$, the product $\gamma(I_1; 1_{W_1}, I_2)$ of I_1 and I_2 exists.*

2. *For any objects W_1, W_2, W_3, W_4 and M_2 in $\mathcal{C}$, any nonzero complex numbers z_0 and z_2 satisfying $|z_2| > |z_0| > 0$, any $P(z_2)$-intertwining map I^1 of type $\binom{W_4}{M_2 W_3}$ and any $P(z_0)$-intertwining map I^2 of type $\binom{M_2}{W_1 W_2}$, the iterate $\gamma(I^1; I^2, 1_{W_3})$ of I^1 and I^2 exists.*

We call either property in Proposition 5.1 the *convergence condition in the category $\mathcal{C}$*. If the convergence condition in the category $\mathcal{C}$ holds, then as images of the product or iterate of certain intertwining maps, $w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$ and $(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}$ for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$ exist as elements of

$$\overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \text{ and } \overline{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3},$$

respectively, when these tensor products exist.

We will use the following concept concerning unique expansion of an analytic function in terms of powers of z and $\log z$: Here we will call a subset $\mathcal{S}$ of $\mathbb{C} \times \mathbb{C}$ a *unique expansion set* if the absolute convergence to 0 on some nonempty open subset of $\mathbb{C}$ of any series $\sum_{(\alpha,\beta)\in\mathcal{S}} a_{\alpha,\beta} z^\alpha (\log z)^\beta$ implies that $a_{\alpha,\beta} = 0$ for all $(\alpha, \beta) \in \mathcal{S}$. It is easy to show that $\mathbb{Z} \times \{0, 1, \ldots, N\}$ is a unique expansion set for any $N \in \mathbb{N}$. More generally, $D \times \{0, 1, \ldots, N\}$ is a unique expansion set for

any discrete subset D of $\mathbb{R}$. On the other hand, it is known that $\mathbb{C} \times \{0\}$ is *not* a unique expansion set[1].

For the rest of this section, we assume for convenience that for any object in $\mathcal{C}$, the (generalized) weights form a discrete set of real numbers. This assumption implies that for any objects W_1, W_2 and W_3 in $\mathcal{C}$, all possible powers (with nonzero coefficients) of x and $\log x$ in the image of any element of $W_1 \otimes W_2$ under any logarithmic intertwining operator of type $\binom{W_3}{W_1\, W_2}$ form a unique expansion set.

Under this assumption, by using (3.7) and (3.15) we can prove the following result:

Proposition 5.2 *Assume that the convergence condition in $\mathcal{C}$ holds. Let z_1, z_2 be two fixed nonzero complex numbers satisfying $|z_1| > |z_2| > 0$, and let I_1 and I_2 be $P(z_1)$- and $P(z_2)$-intertwining maps of types $\binom{W_4}{W_1\, M_1}$ and $\binom{M_1}{W_2\, W_3}$, respectively. Let $w_{(1)} \in W_1$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. Suppose that for any $w_{(2)} \in W_2$,*

$$\langle w'_{(4)}, I_1(w_{(1)}, z_1)I_2(w_{(2)}, z_2)w_{(3)}\rangle = 0.$$

Then

$$\langle w'_{(4)}, \pi_p I_1(w_{(1)}, z_1)\pi_q I_2(w_{(2)}, z_2)w_{(3)}\rangle = 0$$

(recall (2.1)) for all $p, q \in \mathbb{C}$ and $w_{(2)} \in W_2$.

As a consequence, we have:

Corollary 5.3 *Suppose that the $P(z_2)$-tensor product of W_2 and W_3 and the $P(z_1)$-tensor product of W_1 and $W_2 \boxtimes_{P(z_2)} W_3$ both exist. Then the space $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ is spanned as a vector space by the weight components of all elements of the form $w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$, where $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$.*

Analogously, we have the corresponding proposition and corollary concerning iterates of intertwining maps as well, which we will not give here.

Remark 5.4 One can generalize these propositions to the case of products and iterates of more than two intertwining maps. The spanning property in the case of four generalized modules will be used in proving the "coherence theorem" for the constructed "vertex tensor category."

Now suppose that the convergence condition in $\mathcal{C}$ holds. We first give the following definition, which will be important in studying compositions of intertwining maps:

Definition 5.5 Let $z_0, z_1, z_2 \in \mathbb{C}^\times$ with $z_0 = z_1 - z_2$ (so that in particular $z_1 \neq z_2$, $z_0 \neq z_1$ and $z_0 \neq -z_2$). Let (W_1, Y_1), (W_2, Y_2), (W_3, Y_3) and (W_4, Y_4) be generalized V-modules. A $P(z_1, z_2)$-*intertwining map* is a linear map F :

[1] We thank A. Eremenko for informing us of this result.

$W_1 \otimes W_2 \otimes W_3 \to \overline{W}_4$ such that the following conditions are satisfied: the *grading compatibility condition*: for $\beta_i \in \tilde{A}$ and $w_{(i)} \in W_i^{(\beta_i)}$ for $i = 1, 2, 3$,

$$F(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \in \overline{W_4^{(\beta_1+\beta_2+\beta_3)}}; \tag{5.1}$$

the *lower truncation condition*: for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $n \in \mathbb{C}$,

$$\pi_{n-m} F(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = 0 \quad \text{for } \ m \in \mathbb{N} \ \text{ sufficiently large;} \tag{5.2}$$

and the *composite Jacobi identity*:

$$\begin{aligned}
&x_1^{-1}\delta\left(\frac{x_0 - z_1}{x_1}\right) x_2^{-1}\delta\left(\frac{x_0 - z_2}{x_2}\right) Y_4(v, x_0) F(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})\\
&\quad = x_0^{-1}\delta\left(\frac{z_1 + x_1}{x_0}\right) x_2^{-1}\delta\left(\frac{z_0 + x_1}{x_2}\right) F(Y_1(v, x_1) w_{(1)} \otimes w_{(2)} \otimes w_{(3)})\\
&\quad + x_0^{-1}\delta\left(\frac{z_2 + x_2}{x_0}\right) x_1^{-1}\delta\left(\frac{-z_0 + x_2}{x_1}\right) F(w_{(1)} \otimes Y_2(v, x_2) w_{(2)} \otimes w_{(3)})\\
&\quad + x_1^{-1}\delta\left(\frac{-z_1 + x_0}{x_1}\right) x_2^{-1}\delta\left(\frac{-z_2 + x_0}{x_2}\right) F(w_{(1)} \otimes w_{(2)} \otimes Y_3(v, x_0) w_{(3)})
\end{aligned} \tag{5.3}$$

for $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$ (note that the left-hand side of (5.3) is meaningful because any infinite linear combination of the v_n of the form $\sum_{n<N} a_n v_n$ ($a_n \in \mathbb{C}$) acts on any $F(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})$, due to (5.2)).

As in [H1], one shows that in the setting of Proposition 5.1, $\gamma(I_1; 1_{W_1}, I_2)$ is a $P(z_1, z_2)$-intertwining map when $|z_1| > |z_2| > 0$, and $\gamma(I^1; I^2, 1_{W_3})$ is a $P(z_2 + z_0, z_2)$-intertwining map when $|z_2| > |z_0| > 0$.

We now define the following action (cf. Definition 4.8):

Definition 5.6 Let $z_1, z_2 \in \mathbb{C}^\times$, $z_1 \neq z_2$. The linear action $\tau_{P(z_1,z_2)}$ of

$$V \otimes \iota_+\mathbb{C}[t, t^{-1}, (z_1^{-1} - t)^{-1}, (z_2^{-1} - t)^{-1}] \tag{5.4}$$

on $(W_1 \otimes W_2 \otimes W_3)^*$ is defined by

$$\begin{aligned}
&\left(\tau_{P(z_1,z_2)}\left(x_1^{-1}\delta\left(\frac{x_0^{-1} - z_1}{x_1}\right) x_2^{-1}\delta\left(\frac{x_0^{-1} - z_2}{x_2}\right) Y_t(v, x_0)\right)\lambda\right)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})\\
&= \quad x_0\delta\left(\frac{z_1 + x_1}{x_0^{-1}}\right) x_2^{-1}\delta\left(\frac{z_0 + x_1}{x_2}\right) \cdot\\
&\qquad\qquad \lambda(Y_1((-x_0^{-2})^{L(0)} e^{-x_0^{-1}L(1)} v, x_1) w_{(1)} \otimes w_{(2)} \otimes w_{(3)})\\
&\quad + x_0\delta\left(\frac{z_2 + x_2}{x_0^{-1}}\right) x_1^{-1}\delta\left(\frac{-z_0 + x_2}{x_1}\right) \cdot\\
&\qquad\qquad \lambda(w_{(1)} \otimes Y_2((-x_0^{-2})^{L(0)} e^{-x_0^{-1}L(1)} v, x_2) w_{(2)} \otimes w_{(3)})\\
&\quad + x_1^{-1}\delta\left(\frac{-z_1 + x_0^{-1}}{x_1}\right) x_2^{-1}\delta\left(\frac{-z_2 + x_0^{-1}}{x_2}\right) \lambda(w_{(1)} \otimes w_{(2)} \otimes Y_3^o(v, x_0) w_{(3)})
\end{aligned}$$

for $v \in V$, $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$. Also, denote by $Y'_{P(z_1,z_2)}$ the action of $V \otimes \mathbb{C}[t,t^{-1}]$ on $(W_1 \otimes W_2 \otimes W_3)^*$ thus defined, that is,

$$Y'_{P(z_1,z_2)}(v,x) = \tau_{P(z_1,z_2)}(Y_t(v,x)).$$

Remark 5.7 In (14.18) and (14.20) of [H1], actions $\tau^{(1)}_{P(z_1,z_2)}$ and $\tau^{(2)}_{P(z_1,z_2)}$ of (5.4) on $(W_1 \otimes W_2 \otimes W_3)^*$ were defined, when $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively. The action $\tau_{P(z_1,z_2)}$ of (5.4) on $(W_1 \otimes W_2 \otimes W_3)^*$, defined on all of $\{z_1, z_2 \in \mathbb{C}^\times | z_1 \neq z_2\}$, coincides with, and thus extends, these two actions.

Remark 5.8 Using the action $\tau_{P(z_1,z_2)}$, the equality (5.3) can be equivalently written as: For any $w'_{(4)} \in W'_4$,

$$\begin{aligned}
&\left(x_1^{-1}\delta\left(\frac{x_0^{-1}-z_1}{x_1}\right)x_2^{-1}\delta\left(\frac{x_0^{-1}-z_2}{x_2}\right)Y'_4(v,x_0)w'_{(4)}\right)\circ F\\
&\quad = \tau_{P(z_1,z_2)}\left(x_1^{-1}\delta\left(\frac{x_0^{-1}-z_1}{x_1}\right)x_2^{-1}\delta\left(\frac{x_0^{-1}-z_2}{x_2}\right)Y_t(v,x_0)\right)(w'_{(4)}\circ F).
\end{aligned}$$

We will write

$$Y'_{P(z_1,z_2)}(\omega,x) = \sum_{n\in\mathbb{Z}} L'_{P(z_1,z_2)}(n)x^{-n-2}.$$

As in the $P(z)$-intertwining map case, one more notion, corresponding to the lower truncation condition for $P(z_1,z_2)$-intertwining maps, is needed:

Definition 5.9 A map $G \in \mathrm{Hom}(W'_4, (W_1 \otimes W_2 \otimes W_3)^*)$ is said to be *grading restricted* if G respects the $\tilde{A}$-gradings and if given any $n \in \mathbb{C}$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$,

$$G((W'_4)_{[n-m]})(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = 0 \quad \text{for} \;\; m \in \mathbb{N} \;\; \text{sufficiently large.}$$

Clearly, if $G \in \mathrm{Hom}(W'_4, (W_1 \otimes W_2 \otimes W_3)^*)$ is grading restricted, then the map $F \in \mathrm{Hom}(W_1 \otimes W_2 \otimes W_3, \overline{W}_4)$ defined by

$$\langle w'_{(4)}, F(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}\rangle = \langle G(w'_{(4)}), w_{(1)} \otimes w_{(2)} \otimes w_{(3)}\rangle$$

satisfies the grading-compatibility condition (5.1) and the lower truncation condition (5.2).

From these considerations we obtain:

Proposition 5.10 *Let $z_1, z_2 \in \mathbb{C}^\times$, $z_1 \neq z_2$. Let W_1, W_2, W_3 and W_4 be objects in $\mathcal{C}$. Then under the natural map*

$$\mathrm{Hom}(W_1 \otimes W_2 \otimes W_3, \overline{W}_4) \to \mathrm{Hom}(W'_4, (W_1 \otimes W_2 \otimes W_3)^*),$$

the $P(z_1,z_2)$-intertwining maps correspond exactly to the grading restricted maps in $\mathrm{Hom}(W'_4, (W_1 \otimes W_2 \otimes W_3)^*)$ intertwining the actions of the space (5.4) on W'_4 and $(W_1 \otimes W_2 \otimes W_3)^*$.

Let $G \in \mathrm{Hom}(W'_4, (W_1 \otimes W_2 \otimes W_3)^*)$ correspond to a $P(z_1, z_2)$-intertwining map as in Proposition 5.10. Then for any $w'_{(4)} \in W'_4$, $G(w'_{(4)})$ satisfies the following conditions on an element $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$:

The $P(z_1, z_2)$-compatibility condition

(a) The $P(z_1, z_2)$-*lower truncation condition*: For any $v \in V$, the formal Laurent series $Y'_{P(z_1,z_2)}(v, x)\lambda$ involves only finitely many negative powers of x.

(b) The following formula holds: for any $v \in V$,

$$\tau_{P(z_1,z_2)}\left(x_1^{-1}\delta\left(\frac{x_0^{-1}-z_1}{x_1}\right)x_2^{-1}\delta\left(\frac{x_0^{-1}-z_2}{x_2}\right)Y_t(v, x_0)\right)\lambda$$
$$= x_1^{-1}\delta\left(\frac{x_0^{-1}-z_1}{x_1}\right)x_2^{-1}\delta\left(\frac{x_0^{-1}-z_2}{x_2}\right)Y'_{P(z_1,z_2)}(v, x_0)\lambda. \quad (5.5)$$

(Note that the two sides of (5.5) are not *a priori* equal for general $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$.)

The $P(z_1, z_2)$-local grading restriction condition

(a) The *grading condition*: λ is a (finite) sum of generalized eigenvectors in $(W_1 \otimes W_2 \otimes W_3)^*$ for the operator $L'_{P(z_1,z_2)}(0)$ that are also homogeneous with respect to $\tilde{A}$.

(b) The smallest $\tilde{A}$-graded subspace W_λ of $(W_1 \otimes W_2 \otimes W_3)^*$ containing λ and stable under the component operators $\tau_{P(z_1,z_2)}(v \otimes t^m)$ of the operators $Y'_{P(z_1,z_2)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$ has the properties

$$\dim(W_\lambda)^{(\beta)}_{[n]} < \infty, \quad (5.6)$$

$$(W_\lambda)^{(\beta)}_{[n+k]} = 0 \quad \text{for } \; k \in \mathbb{Z} \; \text{ sufficiently negative} \quad (5.7)$$

for any $n \in \mathbb{C}$ and $\beta \in \tilde{A}$, where the subscripts denote the $\mathbb{C}$-grading by $L'_{P(z_1,z_2)}(0)$-eigenvalues and the superscripts denote the natural $\tilde{A}$-grading.

Recall that we have assumed the convergence condition in the category $\mathcal{C}$. We shall now study the condition for the product of two suitable intertwining maps to be written as the iterate of some suitable intertwining maps, and vice versa.

Let z_1, z_2 be distinct nonzero complex numbers, and let $z_0 = z_1 - z_2$. Let $W_!$, W_2, W_3, W_4, M_1 and M_2 be objects in $\mathcal{C}$ and let I_1, I_2, I^1 and I^2 be $P(z_1)$-, $P(z_2)$-, $P(z_2)$- and $P(z_0)$-intertwining maps of types $\binom{W_4}{W_1\, M_1}$, $\binom{M_1}{W_2\, W_3}$, $\binom{W_4}{M_2\, W_3}$ and $\binom{M_2}{W_1\, W_2}$, respectively. Then by the assumption of the convergence condition in $\mathcal{C}$, when $|z_1| > |z_2| > |z_0| > 0$, both $\gamma(I_1; 1_{W_1}, I_2)$ and $\gamma(I^1; I^2, 1_{W_3})$ exist and are $P(z_1, z_2)$-intertwining maps. We shall now discuss when two such $P(z_1, z_2)$-intertwining maps are in fact equal to each other.

As in [H1], we need the following:

Definition 5.11 Let $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$. Define $\mu^{(1)}_{\lambda, w_{(1)}}$ to be the linear functional $\lambda(w_{(1)} \otimes \cdot) \in (W_2 \otimes W_3)^*$ for any $w_{(1)} \in W_1$ and $\mu^{(2)}_{\lambda, w_{(3)}}$ to be $\lambda(\cdot \otimes w_{(3)}) \in (W_1 \otimes W_2)^*$ for any $w_{(3)} \in W_3$.

We have that Lemma 14.3 in [H1] still holds in the generality of the present paper.

As in [H1] the following two conditions on $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ will be needed:

$P^{(1)}(z)$-local grading restriction condition

(a) The $P^{(1)}(z)$-*grading condition*: For any $w_{(1)} \in W_1$, the element $\mu^{(1)}_{\lambda, w_{(1)}} \in (W_2 \otimes W_3)^*$ is the limit, in the locally convex topology defined by the pairing between $(W_2 \otimes W_3)^*$ and $W_2 \otimes W_3$, of an absolutely convergent series of generalized eigenvectors in $(W_2 \otimes W_3)^*$ with respect to the operator $L'_{P(z)}(0)$ that are also homogeneous with respect to $\tilde{A}$.

(b) For any $w_{(1)} \in W_1$, let $W^{(1)}_{\lambda, w_{(1)}}$ be the smallest subspace of $(W_2 \otimes W_3)^*$ containing the terms in the series in (a) and stable under the component operators $\tau_{P(z)}(v \otimes t^m)$ of the operators $Y'_{P(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$. Then $W^{(1)}_{\lambda, w_{(1)}}$ has the properties

$$\dim (W^{(1)}_{\lambda, w_{(1)}})^{(\beta)}_{[n]} < \infty,$$
$$(W^{(1)}_{\lambda, w_{(1)}})^{(\beta)}_{[n+k]} = 0 \quad \text{for} \quad k \in \mathbb{Z} \quad \text{sufficiently negative}$$

for any $n \in \mathbb{C}$ and $\beta \in \tilde{A}$, where the subscripts denote the $\mathbb{C}$-grading by $L'_{P(z)}(0)$-eigenvalues and the superscripts denote the natural $\tilde{A}$-grading.

$P^{(2)}(z)$-local grading restriction condition

(a) The $P^{(2)}(z)$-*grading condition*: For any $w_{(3)} \in W_3$, the element $\mu^{(2)}_{\lambda, w_{(3)}} \in (W_1 \otimes W_2)^*$ is the limit, in the locally convex topology defined by the pairing between $(W_1 \otimes W_2)^*$ and $W_1 \otimes W_2$, of an absolutely convergent series of generalized eigenvectors in $(W_1 \otimes W_2)^*$ with respect to the operator $L'_{P(z)}(0)$ that are also homogeneous with respect to $\tilde{A}$.

(b) For any $w_{(3)} \in W_3$, let $W^{(2)}_{\lambda, w_{(3)}}$ be the smallest subspace of $(W_1 \otimes W_2)^*$ containing the terms in the series in (a) and stable under the component operators $\tau_{P(z)}(v \otimes t^m)$ of the operators $Y'_{P(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$. Then $W^{(2)}_{\lambda, w_{(3)}}$ has the properties

$$\dim (W^{(2)}_{\lambda, w_{(3)}})^{(\beta)}_{[n]} < \infty,$$
$$(W^{(2)}_{\lambda, w_{(3)}})^{(\beta)}_{[n+k]} = 0 \quad \text{for} \quad k \in \mathbb{Z} \quad \text{sufficiently negative}$$

for any $n \in \mathbb{C}$ and $\beta \in \tilde{A}$, where the subscripts denote the $\mathbb{C}$-grading by $L'_{P(z)}(0)$-eigenvalues and the superscripts denote the natural $\tilde{A}$-grading.

By generalizing the arguments in [H1], we can now prove the following two results:

Proposition 5.12 *Let I_1, I_2, I^1 and I^2 be $P(z_1)$-, $P(z_2)$-, $P(z_2)$- and $P(z_0)$-intertwining maps of types $\binom{W_4}{W_1 M_1}$, $\binom{M_1}{W_2 W_3}$, $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively. Then for any $w'_{(4)} \in W'_4$, $\gamma(I_1; 1_{W_1}, I_2)'(w'_{(4)})$ satisfies the $P^{(1)}(z_2)$-local grading restriction condition when $|z_1| > |z_2| > 0$, and $\gamma(I^1; I^2, 1_{W_3})'(w'_{(4)})$ satisfies the $P^{(2)}(z_0)$-local grading restriction condition when $|z_2| > |z_0| > 0$.*

Proposition 5.13 *Let W_1, W_2, W_3, W'_4 and M_1 be objects in $\mathcal{C}$ and assume that $W_1 \boxtimes_{P(z_0)} W_2$ exists in $\mathcal{C}$. Let I_1 and I_2 be $P(z_1)$- and $P(z_2)$-intertwining maps of types $\binom{W_4}{W_1\ M_1}$ and $\binom{M_1}{W_2\ W_3}$, respectively. Suppose that $\gamma(I_1; 1_{W_1}, I_2)'(w'_{(4)})$ satisfies the $P^{(2)}(z_0)$-local grading restriction condition for all $w'_{(4)} \in W'_4$. Then there is a $P(z_2)$-intertwining map of type $\binom{W_4}{W_1 \boxtimes_{P(z_0)} W_2\ W_3}$ such that*

$$\langle w'_{(4)}, I_1(w_{(1)}, z_1) I_2(w_{(2)}, z_2) w_{(3)} \rangle = \langle w'_{(4)}, I(w_{(1)} \boxtimes_{P(z_0)} w_{(2)}, z_2) w_{(3)} \rangle$$

for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. Conversely, let W_1, W_2, W_3, W'_4 and M_2 be objects in $\mathcal{C}$ and assume that $W_2 \boxtimes_{P(z_2)} W_3$ exists in $\mathcal{C}$. Let I^1 and I^2 be $P(z_2)$- and $P(z_0)$-intertwining maps of types $\binom{W_4}{M_2\ W_3}$ and $\binom{M_2}{W_1\ W_2}$, respectively. Suppose that $\gamma(I^1; I^2, 1_{W_3})'(w'_{(4)})$ satisfies the $P^{(1)}(z_2)$-local grading restriction condition for all $w'_{(4)} \in W'_4$. Then there is a $P(z_1)$-intertwining map of type $\binom{W_4}{W_1\ W_2 \boxtimes_{P(z_2)} W_3}$ such that

$$\langle w'_{(4)}, I^1(I^2(w_{(1)}, z_0) w_{(2)}, z_2) w_{(3)} \rangle = \langle w'_{(4)}, I(w_{(1)}, z_1)(w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \rangle$$

for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$.

Proposition 14.7 and Theorem 14.8 in [H1] also hold in the present generality. In particular, Proposition 14.7 in [H1] in our setting states:

Proposition 5.14 *The following two conditions are equivalent:*

1. *For any objects W_1, W_2, W_3, W_4 and M_1 in $\mathcal{C}$, any nonzero complex numbers z_1 and z_2 satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$, any $P(z_1)$-intertwining map I_1 of type $\binom{W_4}{W_1 M_1}$ and $P(z_2)$-intertwining map I_2 of type $\binom{M_1}{W_2 W_3}$ and any $w'_{(4)} \in W'_4$, we have that $\gamma(I_1; 1_{W_1}, I_2)'(w'_{(4)}) \in (W_1 \otimes W_2 \otimes W_3)^*$ satisfies the $P^{(2)}(z_1 - z_2)$-local grading restriction condition.*

2. *For any objects W_1, W_2, W_3, W_4 and M_2 in $\mathcal{C}$, any nonzero complex numbers z_1 and z_2 satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$ and any $P(z_2)$-intertwining map I^1 of type $\binom{W_4}{M_2 W_3}$ and $P(z_1 - z_2)$-intertwining map I^2 of type $\binom{M_2}{W_1 W_2}$ and any $w'_{(4)} \in W'_4$, we have that $\gamma(I^1; I^2, 1_{W_3})(w'_{(4)}) \in (W_1 \otimes W_2 \otimes W_3)^*$ satisfies the $P^{(1)}(z_2)$-local grading restriction condition.*

The proofs are analogous to those in [H1], since the arguments continue to hold in the logarithmic case.

We will call either property in Proposition 5.14 the *expansion condition in the category $\mathcal{C}$*.

Theorem 5.15 *Assume that $\mathcal{C}$ is closed under $P(z)$-tensor products and that the convergence and expansion conditions hold in $\mathcal{C}$. Then there is a unique associativity isomorphism*

$$\alpha_{W_1W_2W_3} : (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \longrightarrow W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$$

for W_1, W_2 and W_3 in $\mathcal{C}$ such that

$$\overline{\alpha}_{W_1W_2W_3}((w_{(1)}\boxtimes_{P(z_1-z_2)}w_{(2)})\boxtimes_{P(z_2)}w_{(3)}) = w_{(1)}\boxtimes_{P(z_1)}(w_{(2)}\boxtimes_{P(z_2)}w_{(3)}) \quad (5.8)$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$, where $\overline{\alpha}_{W_1W_2W_3}$ is the natural extension of $\alpha_{W_1W_2W_3}$ to $\overline{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3}$.

Outline of proof Using the lemmas, propositions and and theorems in this paper in place of those in [H1], we can follow essentially the same steps as in [H1], but treating the logarithmic subtleties with care, to construct these associativity isomorphisms satisfying the condition (5.8). $\square$

Theorem 14.8 in [H1], as generalized to our setting, includes the statement that under the hypotheses of Theorem 5.15, the (general, nonmeromorphic) operator product expansion in the generality of logarithmic intertwining operators exists, and moreover, logarithmic intertwining operators satisfy the canonical associativity property given in Proposition 5.13. If we drop the assumption that $\mathcal{C}$ is closed under $\boxtimes_{P(z)}$, then we still have the existence of a (general, nonmeromorphic) operator product expansion in the generality of logarithmic intertwining operators, in a suitable sense.

6 Differential equations and the convergence and extension properties

For the rest of this paper, for convenience we will take the grading abelian groups A and $\tilde{A}$ to be trivial, so that our (strongly A-graded) conformal vertex algebra V is simply a vertex operator algebra (recall Remark 2.4) and the (strongly graded) generalized V-modules are the generalized V-modules in the sense of Section 2 satisfying the grading restriction conditions referred to in Definition 2.3 and (2.2). We also assume for convenience that for any generalized V-module in $\mathcal{C}$, the (generalized) weights form a discrete set of real numbers.

We first follow [H1] to formulate the convergence and extension properties in the sense of [H1], but now in the logarithmic case, and to assert that these conditions together with algebraic hypotheses imply the expansion condition. We then state results asserting that the theorems on differential equations in

[H2] still hold. This will imply that the condition called C_1-cofiniteness in [H2], together with another algebraic condition, implies that the convergence and extension properties indeed hold.

Given objects W_1, W_2, W_3, W_4, M_1 and M_2 in $\mathcal{C}$, let $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}^1$ and $\mathcal{Y}^2$ be intertwining operators of types $\binom{W_4}{W_1 M_1}$, $\binom{M_1}{W_2 W_3}$, $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively. Consider the following conditions on the product of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ and on the iterate of $\mathcal{Y}^1$ and $\mathcal{Y}^2$, respectively:

Convergence and extension property for products There exists an integer N depending only on $\mathcal{Y}_1$ and $\mathcal{Y}_2$, and for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, $w'_{(4)} \in W'_4$, there exist $M \in \mathbb{N}$, $r_k, s_k \in \mathbb{R}$, $i_k, j_k \in \mathbb{N}$, $k = 1, \dots, M$ and analytic functions $f_{i_k j_k}(z)$ on $|z| < 1$, $k = 1, \dots, M$, satisfying

$$\mathrm{wt}\ w_{(1)} + \mathrm{wt}\ w_{(2)} + s_k > N, \quad k = 1, \dots, M,$$

such that

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_2)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\Big|_{x_1=z_1,\ x_2=z_2}$$

is convergent when $|z_1| > |z_2| > 0$ and can be analytically extended to the multi-valued analytic function

$$\sum_{k=1}^{M} z_2^{r_k}(z_1 - z_2)^{s_k}(\log z_2)^{i_k}(\log(z_1 - z_2))^{j_k} f_{i_k j_k}\left(\frac{z_1 - z_2}{z_2}\right)$$

in the region $|z_2| > |z_1 - z_2| > 0$.

Convergence and extension property for iterates There exists an integer $\tilde{N}$ depending only on $\mathcal{Y}^1$ and $\mathcal{Y}^2$, and for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, $w'_{(4)} \in W'_4$, there exist $\tilde{M} \in \mathbb{N}$, $\tilde{r}_k, \tilde{s}_k \in \mathbb{R}$, $\tilde{i}_k, \tilde{j}_k \in \mathbb{N}$, $k = 1, \dots, \tilde{M}$ and analytic functions $\tilde{f}_{\tilde{i}_k \tilde{j}_k}(z)$ on $|z| < 1$, $k = 1, \dots, \tilde{M}$, satisfying

$$\mathrm{wt}\ w_{(2)} + \mathrm{wt}\ w_{(3)} + \tilde{s}_k > \tilde{N}, \quad k = 1, \dots, \tilde{M},$$

such that

$$\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle_{W_4}\Big|_{x_0=z_1-z_2,\ x_2=z_2}$$

is convergent when $|z_2| > |z_1 - z_2| > 0$ and can be analytically extended to the multi-valued analytic function

$$\sum_{k=1}^{\tilde{M}} z_1^{\tilde{r}_k} z_2^{\tilde{s}_k}(\log z_1)^{\tilde{i}_k}(\log z)^{\tilde{j}_k} \tilde{f}_{\tilde{i}_k \tilde{j}_k}\left(\frac{z_2}{z_1}\right)$$

in the region $|z_1| > |z_2| > 0$.

If for any objects W_1, W_2, W_3, W_4 and M_1 in $\mathcal{C}$ and any intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of the types as above, the convergence and extension property for products holds, we say that *the convergence and extension property for products holds in* $\mathcal{C}$. We similarly define the meaning of the phrase *the convergence and extension property for iterates holds in* $\mathcal{C}$.

In the next theorem *only*, the term "generalized module" will refer to this notion as introduced at the beginning of Section 2, with no grading restriction conditions, that is, without the strong-gradedness assumption (cf. the beginning of Section 4).

Theorem 6.1 *Suppose the following:*

1. *The category $\mathcal{C}$ is closed under $P(z)$-tensor products.*
2. *Every finitely-generated generalized V-module $W = \coprod_{n\in\mathbb{R}} W_{[n]}$ that is lower truncated in the sense that $W_{[n]} = 0$ for n sufficiently negative is an object in $\mathcal{C}$.*
3. *The convergence and extension property for either products or iterates holds in $\mathcal{C}$.*

Then the expansion condition holds in $\mathcal{C}$ (recall Proposition 5.14).

Outline of proof This is proved by analogy with Theorem 16.2 in [H1] and its proof, in particular using Propositions 4.16, 5.12 and 5.14. $\square$

Remark 6.2 As we commented at the end of Section 5, if the first of the three assumptions in Theorem 6.1 is dropped, we still have the existence of a (general, nonmeromorphic) operator product expansion in the generality of logarithmic intertwining operators, in a suitable sense.

Now we discuss generalizations of the results on differential equations in [H2].

For our vertex operator algebra V, let $V_+ = \coprod_{n>0} V_{(n)}$. Let W be a generalized V-module and let $C_1(W) = \mathrm{span}\,\{u_{-1}w \mid u \in V_+,\ w \in W\}$. If $W/C_1(W)$ is finite dimensional, we say that W satisfies the C_1-*cofiniteness condition.* If for any real number N, $\coprod_{n<N} W_{[n]}$ is finite dimensional, we say that W satisfies the *quasi-finite-dimensionality condition.*

Theorem 6.3 *Let $n \geq 1$. For $i = 0, \ldots, n+1$, let W_i be a generalized V-module satisfying the C_1-cofiniteness condition and the quasi-finite-dimensionality condition. Then for any $w_{(i)} \in W_i$ ($i = 0, \ldots, n+1$), there exist $m \geq 1$ and*

$$a_{k,\,l}(z_1, \ldots, z_n) \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}, (z_1 - z_2)^{-1}, (z_1 - z_3)^{-1}, \ldots, (z_{n-1} - z_n)^{-1}]$$

($k = 1, \ldots, m$ and $l = 1, \ldots, n$) such that for any generalized V-modules $\widetilde{W}_i$ ($i = 1, \ldots, n-1$) and any logarithmic intertwining operators $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_{n-1}, \mathcal{Y}_n$

of types $\binom{W_0'}{W_1\widetilde{W}_1}$, $\binom{\widetilde{W}_1}{W_2\widetilde{W}_2}, \dots, \binom{\widetilde{W}_{n-2}}{W_{n-1}\widetilde{W}_{n-1}}$, $\binom{\widetilde{W}_{n-1}}{W_nW_{n+1}}$, *respectively, the (formal logarithmic) series*

$$\langle w_{(0)}, \mathcal{Y}_1(w_{(1)}, z_1)\cdots\mathcal{Y}_n(w_{(n)}, z_n)w_{(n+1)}\rangle \tag{6.1}$$

satisfies the system of differential equations

$$\frac{\partial^m\varphi}{\partial z_l^m} + \sum_{k=1}^{m} a_{k,\, l}(z_1, \dots, z_n)\frac{\partial^{m-k}\varphi}{\partial z_l^{m-k}} = 0, \quad l = 1, \dots, n$$

and is absolutely convergent in the region $|z_1| > \cdots > |z_n| > 0$*; such* $a_{k,\, l}(z_1, \dots, z_n)$ *can be chosen so that the singular points of the system are regular.*

Proof The result on the differential equations in [H2] still holds in our logarithmic case and the proof is essentially the same. □

We say that *products of arbitrary numbers of logarithmic intertwining operators among objects in* $\mathcal{C}$ *are convergent and extendable* if for any $n \geq 1$, any generalized V-modules W_i for $i = 0, \dots, n+1$ and $\widetilde{W}_i$ for $i = 1, \dots, n-1$ in $\mathcal{C}$, and logarithmic intertwining operators $\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_{n-1}, \mathcal{Y}_n$ of types $\binom{W_0'}{W_1\widetilde{W}_1}$, $\binom{\widetilde{W}_1}{W_2\widetilde{W}_2}, \dots, \binom{\widetilde{W}_{n-2}}{W_{n-1}\widetilde{W}_{n-1}}$, $\binom{\widetilde{W}_{n-1}}{W_nW_{n+1}}$, respectively, each (formal logarithmic) series

$$\langle w_{(0)}, \mathcal{Y}_1(w_{(1)}, z_1)\cdots\mathcal{Y}_n(w_{(n)}, z_n)w_{(n+1)}\rangle \tag{6.2}$$

is absolutely convergent in the region $|z_1| > \cdots > |z_n| > 0$ and can be analytically extended to a (multivalued) analytic function in the region given by $z_i \neq 0$ for $i = 1, \dots, n$ and $z_i \neq z_j$ for $i, j = 1, \dots, n$ with $i \neq j$.

We now have:

Theorem 6.4 *Suppose that all generalized* V*-modules in* $\mathcal{C}$ *satisfy the* C_1*-cofiniteness condition and the quasi-finite-dimensionality condition. Then the convergence and extension properties for products and iterates hold in* $\mathcal{C}$ *and in addition, products of arbitrary numbers of logarithmic intertwining operators among objects in* $\mathcal{C}$ *are convergent and extendable.*

Proof This result follows immediately from Theorem 6.3 and the theory of differential equations with regular singular points. □

7 Vertex and braided tensor category structures

Finally, in this section we state the main theorems of our theory. For the notion of vertex tensor category and the connection between these categories and braided tensor categories, see [HL2].

First we have:

Theorem 7.1 *Assume that the hypotheses of Theorem 6.1 hold and in addition that products of arbitrary numbers of logarithmic intertwining operators among objects in $\mathcal{C}$ are convergent and extendable. Then the category $\mathcal{C}$ has a natural structure of vertex tensor category of central charge equal to the central charge c of V such that for each $z \in \mathbb{C}^\times$, the tensor product bifunctor $\boxtimes_{\psi(P(z))}$ associated to $\psi(P(z)) \in \tilde{K}^c(2)$ (see [HL2]) is equal to $\boxtimes_{P(z)}$. In particular, the category $\mathcal{C}$ has a natural braided tensor category structure.*

Outline of proof By Remarks 4.5 and 4.6, we have a tensor product bifunctor for each nonzero complex number. General tensor product bifunctors, associated to elements of the $\frac{c}{2}$-th power of the determinant line bundle over the moduli space of spheres with one negatively oriented puncture and two positively oriented punctures and with local coordinates vanishing at the punctures, are constructed using these bifunctors. The associativity isomorphisms for $P(\cdot)$-tensor products have been constructed in Theorem 5.15. The commutativity isomorphisms for $P(\cdot)$-tensor product functors are constructed using the map Ω_{-1} (recall (3.24) for the logarithmic intertwining operators associated to $P(\cdot)$-tensor products. The general associativity and commutativity isomorphisms are constructed from these special isomorphisms. The unit object is V and the unit isomorphisms are constructed using vertex operators defining the modules. The coherence properties follow easily from the convergence of the products of logarithmic intertwining operators and the characterizations of the associativity, commutativity and unit isomorphisms in terms of tensor products of elements, such as (5.8). $\square$

By Theorem 6.4 we also have the following result, which yields the vertex tensor category structure from purely algebraic hypotheses:

Theorem 7.2 *The conclusion of Theorem 7.1 is still true if the convergence and extension property for products (or iterates) in $\mathcal{C}$, and the condition that products of arbitrary numbers of logarithmic intertwining operators among objects in $\mathcal{C}$ are convergent and extendable, are replaced by the hypothesis that all generalized V-modules in $\mathcal{C}$ are C_1-cofinite and quasi-finite dimensional.*

Let κ be a complex number not in $\mathbb{Q}_{\geq 0}$. Consider the category $\mathcal{O}_\kappa$ of all modules for an affine Lie algebra with central charge $\kappa - h$ (here h is the dual Coxeter number) of finite length, whose composition factors are simple highest weight modules corresponding to weights which are dominant in the direction of the finite-dimensional Lie algebra. Then the theorem above and the results proved by the third author in [Z] give the following:

Theorem 7.3 *The category $\mathcal{O}_\kappa$ has a natural structure of vertex tensor category, and in particular, braided tensor category.*

The braided tensor category structure in this theorem was first constructed by Kazhdan and Lusztig ([KL1]–[KL5]) using a very different method.

References

[B] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* **83** (1986), 3068-3071.

[F] M. Flohr, Bits and pieces in logarithmic conformal field theory, in: *Proceedings of the School and Workshop on Logarithmic Conformal Field Theory and its Applications, Tehran, 2001, Internat. J. Modern Phys.* **A 18** (2003), no. 25, 4497-4591.

[FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, preprint, 1989; *Memoirs Amer. Math. Soc.* **104**, 1993.

[FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Appl. Math., Vol. 134, Academic Press, Boston, 1988.

[Ga] M. Gaberdiel, An algebraic approach to logarithmic conformal field theory, in: *Proceedings of the School and Workshop on Logarithmic Conformal Field Theory and its Applications, Tehran, 2001, Internat. J. Modern Phys.* **A 18** (2003), no. 25, 4593–4638.

[Gu] V. Gurarie, Logarithmic operators in conformal field theory, *Nucl. Phys.* **B410** (1993), 535–549.

[H1] Y.-Z. Huang, A theory of tensor products for module categories for a vertex operator algebra, IV, *J. Pure Appl. Alg.* **100** (1995), 173-216.

[H2] Y.-Z. Huang, Differential equations and intertwining operators, *Communications in Contempoarary Math.* **7** (2005), 375-400.

[HL1] Y.-Z. Huang and J. Lepowsky, Toward a theory of tensor products for representations of a vertex operator algebra, in: *Proc. 20th International Conference on Differential Geometric Methods in Theoretical Physics, New York, 1991*, ed. S. Catto and A. Rocha, World Scientific, Singapore, 1992, Vol. 1, 344–354.

[HL2] Y.-Z. Huang and J. Lepowsky, Tensor products of modules for a vertex operator algebras and vertex tensor categories, in: *Lie Theory and Geometry, in Honor of Bertram Kostant,* ed. R. Brylinski, J.-L. Brylinski, V. Guillemin and V. Kac, Birkhäuser, Boston, 1994, 349–383.

[HL3] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, I, *Selecta Mathematica (New Series)* **1** (1995), 699-756.

[HL4] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, II, *Selecta Mathematica (New Series)* **1** (1995), 757-786.

[HL5] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, III, *J. Pure Appl. Alg.* **100** (1995), 141-171.

[HLLZ] Y.-Z. Huang, J. Lepowsky, H. Li and L. Zhang, On the concepts of intertwining operator and tensor product module in vertex operator algebra theory, *J. Pure Appl. Alg.* **204** (2006), 507-535.

[HLZ] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor product theory for generalized modules for a conformal vertex algebra, to appear.

[KL1] D. Kazhdan and G. Lusztig, Affine Lie algebras and quantum groups, *Duke Math. J., IMRN* **2** (1991), 21–29.

[KL2] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, I, *J. Amer. Math. Soc.* **6** (1993), 905–947.

[KL3] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, II, *J. Amer. Math. Soc.* **6** (1993), 949–1011.

[KL4] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, III, *J. Amer. Math. Soc.* **7** (1994), 335–381.

[KL5] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras, IV, *J. Amer. Math. Soc.* **7** (1994), 383–453.

[LL] J. Lepowsky and H. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Progress in Math., Vol. 227, Birkhäuser, Boston, 2003.

[Lu] D. Lubell, Problem 10992, Problems and Solutions, *American Mathematical Monthly* **110** (2003), 155.

[M] A. Milas, Weak modules and logarithmic intertwining operators for vertex operator algebras, in: *Recent Developments in Infinite-Dimensional Lie Algebras and Conformal Field Theory,* ed: S. Berman, P. Fendley, Y.-Z. Huang, K. Misra, and B. Parshall, Contemporary Math., Vol. 297, American Mathematical Society, Providence, 2002, 201–225.

[RT] M. R. Rahimi Tabar, Disorder systems and logarithmic conformal field theory, in: *Proceedings of the School and Workshop on Logarithmic Conformal Field Theory and its Applications, Tehran, 2001, Internat. J. Modern Phys.* **A 18** (2003), no. 25, 4703–4745.

[Z] L. Zhang, Ph.D. thesis, Rutgers University, 2004.

Department of Mathematics, Rutgers University, Piscataway, NJ 08854
E-mail address: yzhuang@math.rutgers.edu

Department of Mathematics, Rutgers University, Piscataway, NJ 08854
E-mail address: lepowsky@math.rutgers.edu

Department of Mathematics, Rutgers University, Piscataway, NJ 08854
E-mail address: linzhang@math.rutgers.edu